**Problem 1.6** A wave traveling along a string in the $+x$-direction is given by

$$y_1(x,t) = A\cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x,t)$ arrives at the wall, a reflected wave $y_2(x,t)$ is generated. Hence, at any location on the string, the vertical displacement $y_s$ will be the sum of the incident and reflected waves:

$$y_s(x,t) = y_1(x,t) + y_2(x,t).$$

**(a)** Write down an expression for $y_2(x,t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

**(b)** Generate plots of $y_1(x,t)$, $y_2(x,t)$ and $y_s(x,t)$ versus $x$ over the range $-2\lambda \le x \le 0$ at $\omega t = \pi/4$ and at $\omega t = \pi/2$.

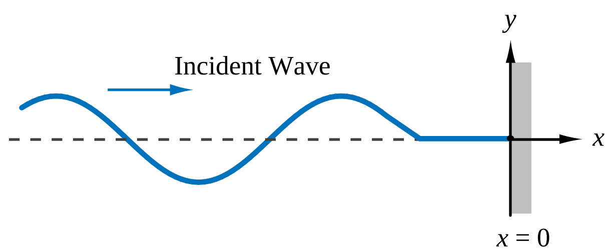


Figure P1.6: Wave on a string tied to a wall at $x = 0$ (Problem 1.6).

**Solution:**

**(a)** Since wave $y_2(x,t)$ was caused by wave $y_1(x,t)$, the two waves must have the same angular frequency $\omega$, and since $y_2(x,t)$ is traveling on the same string as $y_1(x,t)$, the two waves must have the same phase constant $\beta$. Hence, with its direction being in the negative $x$-direction, $y_2(x,t)$ is given by the general form

$$y_2(x,t) = B\cos(\omega t + \beta x + \phi_0), \tag{1}$$

where $B$ and $\phi_0$ are yet-to-be-determined constants. The total displacement is

$$y_s(x,t) = y_1(x,t) + y_2(x,t) = A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $y_s(0,t) = 0$ for all $t$. Thus,

$$y_s(0,t) = A\cos\omega t + B\cos(\omega t + \phi_0) = 0. \tag{2}$$

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = -A$ and $\phi_0 = 0$, in which case we have

$$y_2(x,t) = -A\cos(\omega t + \beta x). \tag{3}$$

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A\cos\omega t + B(\cos\omega t\cos\phi_0 - \sin\omega t\sin\phi_0) = 0,$$

or

$$(A + B\cos\phi_0)\cos\omega t - (B\sin\phi_0)\sin\omega t = 0. \tag{4}$$

This equation has to be satisfied for all values of $t$. At $t = 0$, it gives

$$A + B\cos\phi_0 = 0, \tag{5}$$

and at $\omega t = \pi/2$, (4) gives

$$B\sin\phi_0 = 0. \tag{6}$$

Equations (5) and (6) can be satisfied simultaneously only if

$$A = B = 0 \tag{7}$$

or

$$A = -B \quad \text{and} \quad \phi_0 = 0. \tag{8}$$

Clearly (7) is not an acceptable solution because it means that $y_1(x,t) = 0$, which is contrary to the statement of the problem. The solution given by (8) leads to (3).

**(b)** At $\omega t = \pi/4$,

$$y_1(x,t) = A\cos(\pi/4 - \beta x) = A\cos\left(\frac{\pi}{4} - \frac{2\pi x}{\lambda}\right),$$
$$y_2(x,t) = -A\cos(\omega t + \beta x) = -A\cos\left(\frac{\pi}{4} + \frac{2\pi x}{\lambda}\right).$$

Plots of $y_1$, $y_2$, and $y_3$ are shown in Fig. P1.6(b).

At $\omega t = \pi/2$,

$$y_1(x,t) = A\cos(\pi/2 - \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda},$$
$$y_2(x,t) = -A\cos(\pi/2 + \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda}.$$

Plots of $y_1$, $y_2$, and $y_3$ are shown in Fig. P1.6(c).

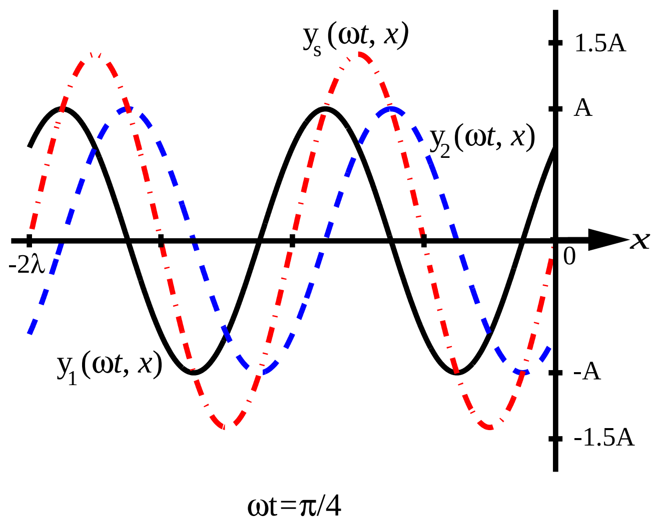


Figure P1.6: (b) Plots of $y_1$, $y_2$, and $y_s$ versus $x$ at $\omega t = \pi/4$.

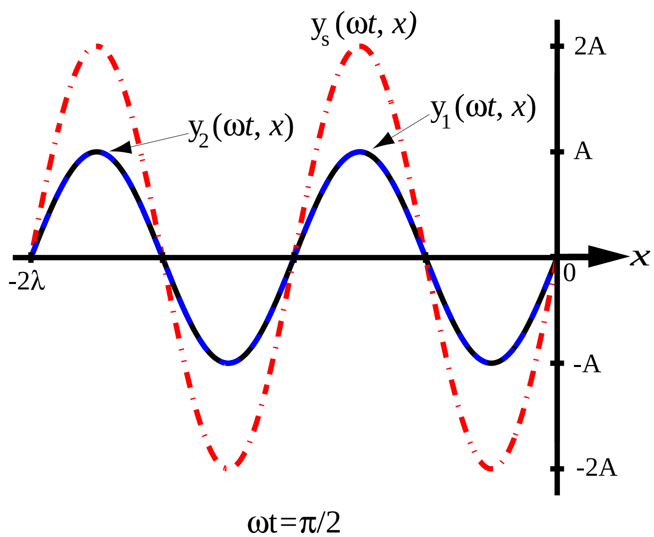


Figure P1.6: (c) Plots of $y_1$, $y_2$, and $y_s$ versus $x$ at $\omega t = \pi/2$.