Problem 6.20 In a certain medium, the direction of current density $\mathbf{J}$ points in the radial direction in cylindrical coordinates and its magnitude is independent of both ϕ and z. Determine $\mathbf{J}$, given that the charge density in the medium is

$$\rho_{\rm v} = \rho_0 r \cos\omega t \quad ({\rm C/m^3}).$$

Solution: Based on the given information,

$$\mathbf{J} = \hat{\mathbf{r}} J_{\rm r}(r).$$

With $J_\phi = J_z = 0$, in cylindrical coordinates the divergence is given by

$$\nabla\cdot\mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_{\rm r}).$$

From Eq. (6.54),

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\rm v}}{\partial t} = -\frac{\partial}{\partial t}(\rho_0 r\cos\omega t) = \rho_0 r\omega\sin\omega t.$$

Hence

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(rJ_{\rm r}) &= \rho_0 r\omega\sin\omega t,\\
\frac{\partial}{\partial r}(rJ_{\rm r}) &= \rho_0 r^2\omega\sin\omega t,\\
\int_0^r \frac{\partial}{\partial r}(rJ_{\rm r})\,dr &= \rho_0\omega\sin\omega t\int_0^r r^2\,dr,\\
rJ_{\rm r}\big|_0^r &= (\rho_0\omega\sin\omega t)\left.\frac{r^3}{3}\right|_0^r,\\
J_{\rm r} &= \frac{\rho_0\omega r^2}{3}\sin\omega t,
\end{aligned}$$

and

$$\mathbf{J} = \hat{\mathbf{r}}J_{\rm r} = \hat{\mathbf{r}}\frac{\rho_0\omega r^2}{3}\sin\omega t \quad ({\rm A/m^2}).$$