Green Forms for Anisotropic, Inhomogeneous Media

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Short Title: Anisotropic Green Forms

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Abstract— The dyadic Green function for an anisotropic, inhomogenous medium is reformulated as a double differential form by embedding material properties in the Hodge star operator. The usual definition of the star operator must be modified in order to allow treatment of media with nonsymmetric permeability or permittivity tensors. This formalism simplifies the manipulation of differential operators in the anisotropic case. An integral equation is obtained which relates the Green form for the electric field to a generalization of the usual isotropic scalar Green function.

1. INTRODUCTION

In this paper we treat the Green form for an anisotropic, inhomogenous, and nonbianisotropic medium. The importance of special cases of this general problem is noted by Moskvin, *et al.* [1], Ren [2] and others. We recover the usual results for electric field due to impressed sources and derive a new integral equation relating the Green form to a generalization of the usual scalar Green function for isotropic media.

In a previous paper [3], we treated the reformulation of the dyadic Green function for isotropic media as a double differential form [4]. Here we treat the anisotropic, inhomogeneous case by embedding material properties into the Hodge star operator. Such an approach is suggested by Bamberg and Sternberg [5]. The usual definition of the Hodge star operator must be modified slightly in order to allow the treatment of nonreciprocal or lossy media. This formalism simplifies the derivation of the anisotropic generalization of Huygens' principle and other results as compared to the usual dyadic notation, due to the use of the product rule for the exterior derivative, the generalized Stokes theorem, and other identities of the calculus of differential forms.

For an isotropic medium, the Green form for the electric field can be obtained in terms of the scalar Green function of the Helmholtz wave equation [3]. We obtain a generalization of this relationship which is valid for anisotropic, inhomogeneous media. The scalar Helmholtz equation extends in a natural way to an anisotropic, inhomogeneous medium, and the Green form for this anisotropic Helmholtz equation is a generalization of the scalar Green function for an isotropic medium. The anisotropic Helmholtz Green form is related to the Green form for the electric field by an integral equation.

By specializing to a homogeneous medium, the integral equation and definitions for the Green forms can be transformed to the wavenumber representation, leading to expressions for the Green form for the electric field and the Fresnel equation valid for nonreciprocal or lossy media. For a homogeneous medium, the Helmholtz Green form can be obtained exactly in physical space, so that the integral equation leads to a new representation for the Green form for the electric field. For a biaxial medium, the series solution for the integral equation relating the two Green forms can also be resummed in the wavevector representation, yielding an expression similar in form to the usual relationship between the Green form and the scalar Green function for an isotropic medium.

We employ a suppressed $e^{-i\omega t}$ time dependence throughout this paper.

2. THE GREEN DOUBLE FORM

We consider in this paper electromagnetic propagation in a nonbianisotropic medium with macroscopic electromagnetic properties characterized by invertible permittivity and permeability tensors $\epsilon_{ij}(\mathbf{r})$ and $\mu_{ij}(\mathbf{r})$. Maxwell's laws are

$$dE = i\omega B \tag{1a}$$

$$dH = -i\omega D + J \tag{1b}$$

$$dD = \rho \tag{1c}$$

$$dB = 0 \tag{1d}$$

where E and H are the electric and magnetic field intensity 1–forms, D and B are the electric and magnetic flux density 2–forms, J is the electric current density 2–form, and ρ is the electric charge density 3–form.

In order to represent the constitutive relations on differential forms, we define the Hodge star operators $\star_e$ and $\star_h$ using the permittivity and permeability tensors respectively, as described in the Appendix. The constitutive relations can then be written

$$D = \star_e E \tag{2a}$$

$$B = \star_h H. \tag{2b}$$

In rectangular coordinates,

$$\begin{aligned}\star_e(E_1\,dx + E_2\,dy + E_3\,dz) = \; & (\epsilon_{11}E_1 + \epsilon_{12}E_2 + \epsilon_{13}E_3)\,dy\,dz + \\ & (\epsilon_{21}E_1 + \epsilon_{22}E_2 + \epsilon_{23}E_3)\,dz\,dx + \\ & (\epsilon_{31}E_1 + \epsilon_{32}E_2 + \epsilon_{33}E_3)\,dx\,dy\end{aligned} \tag{3}$$

where for conciseness the wedges denoting the exterior product between differentials are omitted. If the star operator $\star_e$ is applied to a 2–form,

$$\begin{aligned}\star_e(D_1\,dy\,dz + D_2\,dz\,dx + D_3\,dx\,dy) = \; & (\epsilon^{11}D_1 + \epsilon^{12}D_2 + \epsilon^{13}D_3)\,dx + \\ & (\epsilon^{21}D_1 + \epsilon^{22}D_2 + \epsilon^{23}D_3)\,dy + \\ & (\epsilon^{31}D_1 + \epsilon^{32}D_2 + \epsilon^{33}D_3)\,dz\end{aligned} \tag{4}$$

where the ϵ^{ij} are components of ϵ^{-1}. On 1–forms and 3–forms,

$$\star_e 1 = (\det \epsilon_{ij})\,dx\,dy\,dz \tag{5}$$

The magnetic star operator $\star_h$ behaves similarly. As discussed in the Appendix, the inverses of these star operators for nonreciprocal media are $\star_e{}^{-1} = \tilde{\star}_e$ and $\star_h{}^{-1} = \tilde{\star}_h$, where the tilde denotes transposition of the metric tensor in the definition of the star operator. For reciprocal media characterized by symmetric permittivity and permeability tensors, $\star_e = \tilde{\star}_e$ and $\star_h = \tilde{\star}_h$.

From Maxwell's laws and the constitutive relations, it follows that

$$(-\star_h d\tilde{\star}_h d + \omega^2 \star_h \star_e)E = -i\omega \star_h J. \tag{6}$$

The Green double $1 \otimes 1$ form G for the differential equation (6) is given by

$$(-\star_h d\tilde{\star}_h d + \omega^2 \star_h \star_e) G(\mathbf{r}_1, \mathbf{r}_2) = -\star_h \delta(\mathbf{r}_1 - \mathbf{r}_2) I. \tag{7}$$

where I is the unit $2 \otimes 1$ form $dy_1\, dz_1 \otimes dx_2 + dz_1\, dx_1 \otimes dy_2 + dx_1\, dy_1 \otimes dz_2$ and $\otimes$ denotes the tensor product. Operators act on the $\mathbf{r}_1$ coordinates unless otherwise noted or there is no ambiguity in an expression. The star operators are in general functions of position, and they are assumed to be evaluated at the coordinate of the differentials on which they operate. In rectangular coordinates, the Green form G has components

$$G_{ij}(\mathbf{r}_1, \mathbf{r}_2)\, dx_1^i \otimes dx_2^j$$

where the superscripts index the coordinates dx, dy, and dz.

In order to treat the general, nonreciprocal case, we define the transpose of G to be the $1 \otimes 1$ double form $\tilde{G}$ satisfying

$$(-\tilde{\star}_h d \star_h d + \omega^2 \tilde{\star}_h \tilde{\star}_e) \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) = -\tilde{\star}_h \delta(\mathbf{r}_1 - \mathbf{r}_2) I. \tag{8}$$

Our definition for $\tilde{G}$ differs from Chew's [6] definition for the dyadic Green function for an anisotropic, inhomogeneous medium due to the presence of an additional factor of $\tilde{\star}_h$ on the left–hand side of (8).

Let L and $\tilde{\mathrm{L}}$ be the differential operators of Eqs. (7) and (8) respectively. In order to obtain the electric field in terms of $\tilde{G}$, the operators L and $\tilde{\mathrm{L}}$ must be such that a relationship of the form

$$E_1 \wedge (\tilde{\star}_h \mathrm{L} E_2) - E_2 \wedge (\star_h \tilde{\mathrm{L}} E_1) = dP \tag{9}$$

holds for arbitrary E_1 and E_2. This approach follows the usual formal theory of the Green function for an arbitrary operator, in which conditions for symmetric and self–adjoint Green functions are easily related to properties of the differential operator and boundary conditions. This allows relationships between G and $\tilde{G}$ for reciprocal and lossless media to be conveniently obtained in Sec. 2.2.

The product rule for the exterior derivative [5], $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{deg\, \alpha} \alpha \wedge d\beta$, and the relationship $\nu \wedge \star\lambda = (-1)^{p(n-p)} \tilde{\star}\nu \wedge \lambda$ for p–forms ν and λ obtained in the Appendix can be used to show that

$$d(E_1 \wedge \tilde{\star}_h dE_2 + \star_h dE_1 \wedge E_2) = d\star_h dE_1 \wedge E_2 - E_1 \wedge d\tilde{\star}_h dE_2. \tag{10}$$

Applying this identity to (9) yields

$$P = E_1 \wedge \tilde{\star}_h dE_2 + \star_h dE_1 \wedge E_2 \tag{11}$$

for the boundary term. Note that star operators cannot be moved across the exterior products in this expression since E_1 and E_2 do not have the same degree as dE_2 and dE_1.

Integrating Eq. (9) over a volume V_1 and applying the generalized Stokes theorem

$$\int_{\partial V} \omega = \int_V d\omega \tag{12}$$

yields a generalization of Green's theorem for the operators L and $\tilde{\mathrm{L}}$,

$$\int_{V_1} E_1 \wedge (\tilde{\star}_h \mathrm{L} E_2) - \int_{V_1} E_2 \wedge (\star_h \tilde{\mathrm{L}} E_1) = \int_{\partial V_1} P \tag{13}$$

where ∂V_1 denotes the boundary of V_1. Replacing $E_1(\mathbf{r_1})$ with $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$ in (13) and using the definition (8) produces the generalized Huygens principle for anisotropic, inhomogeneous media,

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge J(\mathbf{r_1}) + \int_{\partial V_1} \left[i\omega \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge H(\mathbf{r}_1) + \star_h d\tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) \right] \tag{14}$$

This is equivalent to the dyadic result given in [6]. Note the absence of surface normal vectors in (14); normal E and H fields naturally do not contribute to the surface integration.

The corresponding dyadic derivation requires the use of vector identities which with differential forms are immediate consequences of the product rule for the exterior derivative and the generalized Stokes theorem. When the calculus of differential forms is employed, manipulations rely on basic properties of the exterior derivative and the star operator rather than tabulated identities. Further simplification results from the fact that the anisotropic star operator has nearly the same properties as the isotropic star operator. For these reasons, differential forms are ideal for coordinate–free manipulations such as those performed in this paper.

2.1 Boundary Conditions

In this section, we seek to determine boundary conditions on E_1 and E_2 such that the surface term on the right–hand side of (13) vanishes. If the fields satisfy boundary conditions such that the surface term vanishes, then replacing $E_1(\mathbf{r}_1)$ with $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$ and $E_2(\mathbf{r}_1)$ with $G(\mathbf{r}_1, \mathbf{r}_3)$ in (13) shows that

$$\tilde{G}(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3). \tag{15}$$

Thus, for properly chosen boundary conditions, G is the transpose of $\tilde{G}$. We discuss radiation boundary conditions, Neumann (magnetically conducting), and Dirichlet (electrically conducting) boundary conditions.

Since the operator L is not in general self–adjoint, E_1 and E_2 will satisfy different boundary conditions. The condition for E_1 will be related to that for E_2 by transposition of star operators. Suppose that E_1 and H_1 have the asymptotic behavior

$$\lim_{r \to \infty} r \left[H_1 - \star_h \frac{k}{\omega} dr \wedge E_1 \right] = 0 \tag{16}$$

and E_2 and H_2 satisfy the transposed condition

$$\lim_{r\to\infty} r\left[H_2 - \tilde{\star}_h \frac{k}{\omega}\, dr \wedge E_2\right] = 0 \tag{17}$$

where we assume that the medium is such that outgoing waves have r^{-1} dependence and that $k(\mathbf{r})$ is determined by the particular form of the outgoing waves. Using Faraday's law and the constitutive relation $B = \star_h H$, we obtain $\tilde{\star}_h dE_1 = i\omega H_1$. By the transpose of this relationship, $\star_h dE_2 = i\omega H_2$. Thus, P can be put into the form

$$P = i\omega(E_1 \wedge H_2 + H_1 \wedge E_2). \tag{18}$$

Let V_1 is a sphere with radius r. The surface integral term of (13) is then

$$\begin{aligned}\int_{\partial V} P &= \int_{\partial V} i\omega[E_1 \wedge (\tilde{\star}_h \frac{k}{\omega}\, dr \wedge E_2) + H_1 \wedge E_2] \\ &= \int_{\partial V} i\omega[-\frac{k}{\omega}\star_h(\mathbf{r})\, dr \wedge E_1 + H_1] \wedge E_2\end{aligned}$$

which vanishes by the condition (17).

For electrically conducting boundary conditions, the 1-forms E_1 and E_2 will be oriented perpendicular to the boundary, so that if n is a coordinate normal to the boundary, E_1 and E_2 are proportional to dn. The 2-forms $E_1 \wedge H_2$ and $H_1 \wedge E_2$ therefore must each contain a factor of dn. Since the surface integration is over all coordinates except n, the boundary term of (13) vanishes and Eq. (15) holds. The surface term also vanishes for magnetically conducting boundary conditions by the same reasoning.

2.2 Symmetry and Self-Adjointness Conditions

The conditions on the permeability and permittivity tensors for reciprocal and lossless media can be related to the symmetry and self–adjointness of G. For a reciprocal medium, $\star_h = \tilde{\star}_h$ and $\star_e = \tilde{\star}_e$. By definition, in order for the operator $\star_h L$ to be symmetric with respect to the reaction inner product

$$< E, J >= \int_V E \wedge J \tag{19}$$

where E is a 1-form and J is a 2-form, we must have

$$\int_V E_1 \wedge (\star_h \mathrm{L} E_2) = \int_V E_2 \wedge (\star_h \mathrm{L} E_1). \tag{20}$$

By Ampere's and Faraday's laws, (20) is equivalent to

$$\int_V E_1 \wedge J_2 = \int_V E_2 \wedge J_1 \tag{21}$$

which vanishes by the definition of reciprocity. Thus, for a reciprocal medium, the operator L is symmetric. (We assume that E_1 and E_2 satisfy the same boundary condition, since an operator may be equal to its transpose but still not represent a symmetric boundary value problem unless the operator and its transpose also operate on the same domain of a function space.) The derivation of (13) shows that the equality (20) leads to the Lorentz reciprocity theorem

$$\int_{\partial V}(E_1 \wedge H_2 + H_1 \wedge E_2) = 0. \tag{22}$$

for a medium with symmetric but otherwise arbitrary permittivity and permeability tensors. Replacing E_1 with $G(\mathbf{r}_1, \mathbf{r}_2)$ and E_2 with $G(\mathbf{r}_1, \mathbf{r}_3)$ in Eq. (20) gives the reciprocity relation [6]

$$G(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3) \tag{23}$$

for a medium with symmetric permittivity and permeability tensors.

A lossless medium is characterized by hermitian permittivity and permeability tensors, so that $\star_e = \tilde{\star}_e{}^*$ and $\star_h = \tilde{\star}_h{}^*$, where the superscript * denotes complex conjugation of the coefficients of the permittivity and permeability tensors in the definition of the star operator. If $\tilde{\star}_h L$ is self-adjoint with respect to the inner product

$$< E, J >= \int_V E^* \wedge J \tag{24}$$

we must have that

$$\int_V E_1^* \wedge (\tilde{\star}_h \mathrm{L} E_2) = \left[\int_V E_2^* \wedge (\tilde{\star}_h \mathrm{L} E_1)\right]^*. \tag{25}$$

Using (6), this is equivalent to

$$\int_V E_1^* \wedge J_2 = \int_V E_2 \wedge J_1^*. \tag{26}$$

Using a modification of the derivation of (13), we obtain

$$\int_V E_1^* \wedge (\tilde{\star}_h \mathrm{L} E_2) - \left[\int_V E_2^* \wedge (\tilde{\star}_h \mathrm{L} E_1)\right]^* = i\omega \int_{\partial V}(E_1^* \wedge H_2 + H_1^* \wedge E_2). \tag{27}$$

The surface integral term on the right-hand side vanishes for a properly chosen boundary condition on E_1 and E_2, including any of those discussed in Sec. 2.1, so that G is self-adjoint with respect to the inner product (24). Replacing E_1 with $G(\mathbf{r}_1, \mathbf{r}_2)$ and E_2 with $G(\mathbf{r}_1, \mathbf{r}_3)$ in Eq. (25) then yields

$$G^*(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3) \tag{28}$$

for a lossless medium and fields satisfying appropriate boundary conditions.

3. GREEN FORM FOR THE ANISOTROPIC HELMHOLTZ EQUATION

In general, the operator L of Eq. (6) is not diagonal. This makes the Green form G difficult to obtain in physical space. Adding $d\star_h d\star_h E$ to both sides of Eq. (6) yields

$$(\Delta_h + \omega^2 \star_h \star_e)E = -i\omega \star_h J + d\star_h d\tilde{\star}_h E \tag{29}$$

where Δ_h is the wave operator in the metric due to the permeability of the medium and is defined in the Appendix. The operator Δ_h is diagonal, so that the inverse g of the operator $\Delta_h + \omega^2 \star_h \star_e$ can be obtained in some cases for which no analytic solution for G is known. We therefore define the Green $1 \otimes 1$ form g for the anisotropic Helmholtz equation,

$$(\Delta_h + \omega^2 \star_h \star_e) g(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) I \tag{30}$$

where operators act on the r_1 coordinate and I is the unit $1 \otimes 1$ form. In an isotropic medium, $g = \mu_0^2 g_0 I$, where $g_0 = e^{ik_0 r}/(4\pi r)$ is the usual scalar Green function.

Note that in Eq. (30) we have not included the $\star_h$ operator on the right–hand side as was done in Eq. (7). With the Helmholtz Green form, we modify the form of the definitions of G and $\tilde{G}$ as well as the definition of the inner product used in the Green theorem (13) in order to make use of the symmetry of Δ_h. Thus, the adjoint $2 \otimes 1$ Green form $\tilde{g}$ is defined to be

$$(\Delta_h + \omega^2 \tilde{\star}_e \tilde{\star}_h) \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) I \tag{31}$$

where I is the unit $2 \otimes 1$ form. We seek to obtain a relationship of the form

$$C_1 \wedge \mathrm{M} E_2 - E_2 \wedge \tilde{\mathrm{M}} C_1 = dQ \tag{32}$$

where E_1 is an arbitrary 1-form, C_1 is an arbitrary 2-form, and M and $\tilde{\mathrm{M}}$ are the differential operators used in the definitions of g and $\tilde{g}$ respectively. The surface term can be shown to be

$$Q = \tilde{\star}_h C_1 \wedge \tilde{\star}_h dE_2 + \star_h d\tilde{\star}_h C_1 \wedge E_2 + C_1 \wedge \tilde{\star}_h d\star_h E_2 - \star_h dC_1 \wedge \star_h E_2. \tag{33}$$

From (9) we have that

$$\int_V C_1 \wedge \mathrm{M} E_2 - \int_V E_2 \wedge \tilde{\mathrm{M}} C_1 = \int_{\partial V} Q. \tag{34}$$

Substituting $\tilde{g}(\mathbf{r}_1, \mathbf{r}_2)$ for $C_1(\mathbf{r})$ and using Eqs. (29) and (31),

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge d\tilde{\star}_h d\star_h E(\mathbf{r}_1) + \int_{\partial V_1} R \tag{35}$$

where R is

$$\begin{aligned} R \;=\; & \tilde{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \tilde{\star}_h dE(\mathbf{r}_1) + \star_h d\tilde{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) \\ & + \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \tilde{\star}_h d\star_h E(\mathbf{r}_1) - \star_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h E(\mathbf{r}_1) \end{aligned} \tag{36}$$

Integrating the second term on the right–hand side of (35) twice by parts cancels two of the terms of R, leaving

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{\star}_h d\tilde{\star}_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) + \int_{\partial V_1} R_1 \tag{37}$$

where the operator $\tilde{\star}_h d\tilde{\star}_h d$ acts on the $\mathbf{r}_1$ part of $\tilde{g}$ and

$$R_1 = \tilde{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \tilde{\star}_h dE(\mathbf{r}_1) + \star_h d\tilde{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1). \tag{38}$$

Assuming that $\tilde{\star}_h \tilde{g}$ and E satisfy boundary conditions such that the surface integral term vanishes, Eq. (37) reduces to

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{\star}_h d\tilde{\star}_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) \tag{39}$$

which is an integral equation for E in terms of the source J and Helmholtz Green form $\tilde{g}$. It may be possible to employ this equation as a basis for numerical techniques for treating propagation and scattering problems in anisotropic media.

The integral equation (39) is valid for an arbitrarily anisotropic, inhomogeneous medium. Two special cases are of greatest interest. For a homogeneous, anisotropic medium, the Helmholtz Green form can be found exactly, as will be shown in the next section. For an electrically inhomogeneous, isotropic medium, finding the Helmholtz Green form (for some boundary condition) reduces to the determination of the Green function g_s for the scalar wave equation

$$[\Delta + k^2(\mathbf{r})]u(\mathbf{r}) = f(\mathbf{r}) \tag{40}$$

where $u(\mathbf{r})$ is a function. The Helmholtz Green form is then $\mu_0^2 g_s I$. Thus, Eq. (39) connects scalar scattering for an electrically inhomogeneous, isotropic medium with the full scattering problem for the electric field E.

3.1 Integral Relationship Between G and $\tilde{g}$

Substituting $\tilde{g}(\mathbf{r}_1, \mathbf{r}_2)$ for $C_1(\mathbf{r})$ and $G(\mathbf{r}_1, \mathbf{r}_3)$ for E_2 in Eq. (34) and following a procedure similar to the derivation of (39), we obtain the integral equation

$$G(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\star}_h \tilde{g}(\mathbf{r}_2, \mathbf{r}_1) - \int_{V_3} \tilde{\star}_h d\tilde{\star}_h d\tilde{g}(\mathbf{r}_3, \mathbf{r}_1) \wedge G(\mathbf{r}_3, \mathbf{r}_2). \tag{41}$$

which is similar in form to Dyson's equation. This relationship generalizes the usual relationship between the scalar Green function and the Green form for isotropic media [3]. By repeated substitution on (41) we obtain the formal series solution for G,

$$\tilde{G} = \tilde{\star}_h \tilde{g} - \int \tilde{\star}_h d\tilde{\star}_h d\tilde{g} \wedge \tilde{\star}_h \tilde{g} + \int\int \tilde{\star}_h d\tilde{\star}_h d\tilde{g} \wedge \tilde{\star}_h d\tilde{\star}_h d\tilde{g} \wedge \tilde{\star}_h \tilde{g} \; - \; \cdots \tag{42}$$

The series is divergent; each term after the first can be thought of as the field due to an infinite volume current density. For a biaxial medium, however, the wavevector representation of this series can be resummed.

3.2 Symmetric Permeability Tensor

For a symmetric permeability tensor, we can simplify expressions (39) and (41) by using the modified definitions

$$(\Delta_h + \omega^2 \star_h \star_e) g = -\star_h \delta I \tag{43a}$$

$$(\Delta_h + \omega^2 \star_h \tilde{\star}_e) \tilde{g} = -\star_h \delta I \tag{43b}$$

where g and $\tilde{g}$ are $1 \otimes 1$ forms and I is the unit $2 \otimes 1$ form. For a symmetric star operator, we can obtain an identity of the form

$$E_1 \wedge \star_h \mathrm{M}' E_2 - E_2 \wedge \star_h \tilde{\mathrm{M}}' E_1 = dQ' \tag{44}$$

which replaces Eq. (32).

One can show that with the modified definition of $\tilde{g}$, Eq. (39) simplifies to

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge J(\mathbf{r}_1) - \int_{V_1} \star_h d \star_h d \star_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) \tag{45}$$

The integral equation (41) becomes

$$\tilde{G}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) - \int_{V_3} \star_h d \star_h d \star_h \tilde{g}(\mathbf{r}_3, \mathbf{r}_2) \wedge \tilde{G}(\mathbf{r}_1, \mathbf{r}_3) \tag{46}$$

for a symmetric $\star_h$ operator and the modified definitions (43).

It is interesting to compare (45) to the expression for E obtained from the usual relationship [3] between G and g for an isotropic medium. For an isotropic medium, Eq. (45) becomes

$$E = i\omega\mu_0 \int g \wedge J - \int \star d \star d \star g \wedge E \tag{47}$$

where g has been scaled by μ_0^2 to agree with the standard definition for the scalar green function. The usual expression for the G in terms of g yields

$$E = i\omega\mu_0 \int g \wedge J - \int \star d \star d \star g \wedge \left(-\frac{i}{\omega\epsilon} \star J \right) \tag{48}$$

for the electric field. These two expressions are identical except for the replacement of E with $-i/(\omega\epsilon) \star J$.

4. HOMOGENEOUS MEDIA

For a homogeneous medium, by spatial symmetry the components of g are shift-invariant functions $g_{ij}(\mathbf{r}_2 - \mathbf{r}_1)$, and the integral in (41) is a convolution. The spatial Fourier transform of Eq. (41) is then

$$G = \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \tilde{g}^T \mathbf{k}\mathbf{k}^T \mu G \tag{49}$$

where $\mathbf{k}$ is the wavevector. In this and later expressions, G and $\tilde{g}$ denote the matrices of the Fourier transforms of the coefficients of the double forms. Solving for G,

$$G = \left[\mu \tilde{g}^{T-1} - \frac{1}{\det \mu} \mu \mathbf{k}\mathbf{k}^T \mu \right]^{-1} . \tag{50}$$

The Fourier transform of Eq. (31) shows that

$$\tilde{g} = \left[\frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) I - \omega^2 \epsilon^T \mu^{T-1} \right]^{-1} \tag{51}$$

where I is the identity matrix. Eq. (7) gives an alternate expression for G,

$$G = \left[-\Gamma \mu^{T-1} \Gamma - \omega^2 \epsilon \right]^{-1} \tag{52}$$

where

$$\Gamma = \begin{bmatrix} 0 & -k_z & k_y \\ \cdot k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} . \tag{53}$$

Substituting (51) into (50) gives

$$G = \left[-\frac{1}{\det \mu} \mu \mathbf{k}\mathbf{k}^T \mu + \frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) \mu - \omega^2 \epsilon \right]^{-1} \tag{54}$$

which is equivalent to the result obtained in [7]. The poles of G dominate the inverse Fourier transform in the far field, so that

$$\det \left[-\frac{1}{\det \mu} \mathbf{k}\mathbf{k}^T \mu + \frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) I - \omega^2 \mu^{-1} \epsilon \right] = 0 \tag{55}$$

is the Fresnel equation [7, 8] for a medium with homogeneous but otherwise arbitrary (invertible) permittivity and permeability tensors.

The definition (80) of the wave operator Δ_h for a homogeneous medium yields the identity

$$-\frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) I = \mu^{-1} \Gamma \mu^{T-1} \Gamma - \frac{1}{\det \mu} \mathbf{k}\mathbf{k}^T \mu \tag{56}$$

when Fourier transformed. This expression shows the utility of Δ_h, which has diagonal wavevector representation for an arbitrary (homogeneous) permeability tensor.

If μ_{ij} is diagonalizable by a rotation, then the inverse transform of $\tilde{g}(\mathbf{k})$ can be obtained. From Eq. (51),

$$\tilde{g}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \left[\frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) I - \omega^2 \epsilon^T \mu^{T^{-1}}\right]^{-1} \tag{57}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Rotating coordinates so that μ_{ij} is diagonal and performing a change of variables, this becomes

$$\tilde{g}(\mathbf{r}) = \frac{1}{(2\pi)^3} \frac{(\det \mu)^{3/2}}{\sqrt{\mu_1 \mu_2 \mu_3}} \int d\mathbf{k}' e^{i\sqrt{\det \mu}(k'_x x/\sqrt{\mu_1} + k'_y y/\sqrt{\mu_2} + k'_z z/\sqrt{\mu_3})} \left[k'^2 I - \omega^2 \epsilon^T \mu^{T^{-1}}\right]^{-1} \tag{58}$$

where μ_1, μ_2, and μ_3 are the eigenvalues of μ_{ij} and x, y, and z are the components of $\mathbf{r}$. Let $\tilde{\mathbf{r}} = \sqrt{\det \mu}(\hat{\mathbf{x}} x/\sqrt{\mu_1} + \hat{\mathbf{y}} y/\sqrt{\mu_2} + \hat{\mathbf{z}} z/\sqrt{\mu_3})$. By rotating $\mathbf{k}'$ so that k'_z is in the $\tilde{\mathbf{r}}$ direction,

$$\tilde{g}(\mathbf{r}) = \frac{\det \mu}{(2\pi)^3} \int k'^2 \sin\theta \, dk' \, d\theta \, d\phi \, e^{ik'\tilde{r}\cos\theta} \left[k'^2 I - \omega^2 \epsilon^T \mu^{T^{-1}}\right]^{-1}. \tag{59}$$

Integrating the angles,

$$\tilde{g}(\mathbf{r}) = \frac{\det \mu}{4i\pi^2 \tilde{r}} \int k' dk' \left(e^{ik'\tilde{r}} - e^{-ik'\tilde{r}}\right) \left[k'^2 I - \omega^2 \epsilon^T \mu^{T^{-1}}\right]^{-1}. \tag{60}$$

This integral can be performed if $\epsilon^T \mu^{T^{-1}}$ has a square root.

The matrix $\epsilon^T \mu^{T^{-1}}$ is not in general diagonalizable [7], but it has a Jordan normal form SJS^{-1}. Consider one of the Jordan blocks of J, with $\lambda = re^{i\alpha}$. For this block, we construct the square root

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}^{1/2} = \begin{bmatrix} \pm\sqrt{r}e^{i\alpha/2} & \pm 1/(2\sqrt{r}e^{i\alpha/2}) & & \\ & \ddots & \ddots & \\ & & \ddots & +1/(2\sqrt{r}e^{i\alpha/2}) \\ & & & \pm\sqrt{r}e^{i\alpha/2} \end{bmatrix} \tag{61}$$

where the sign is chosen so that $\mathrm{Re}\{\pm\sqrt{r}e^{i\alpha/2}\}$ is positive; the other root can be discarded by causality. The right-hand side of (61) exists since ϵ_{ij} and μ_{ij} are by assumption invertible, so that $\epsilon^T \mu^{T^{-1}}$ has no zero eigenvalues. In this manner, we can construct $J^{1/2}$, so that $K = \omega S J^{1/2} S^{-1}$ is a square root of $\omega^2 \epsilon^T \mu^{T^{-1}}$. Equation (60) then becomes

$$\tilde{g}(\mathbf{r}) = \frac{\det \mu}{8i\pi^2 \tilde{r}} \int_0^\infty dk' \left(e^{ik'\tilde{r}} - e^{-ik'\tilde{r}}\right) \left[(k' - K)^{-1} + (k' + K)^{-1}\right]. \tag{62}$$

This can be rewritten as

$$\tilde{g}(\mathbf{r}) = \frac{\det \mu}{8i\pi^2 \tilde{r}} \int_{-\infty}^\infty dk' \left[e^{ik'\tilde{r}} (k' - K)^{-1} - e^{-ik'\tilde{r}} (k' - K)^{-1}\right]. \tag{63}$$

By causality, the second term can be discarded, and the final result is

$$\tilde{g}(\mathbf{r}) = (\det \mu) \frac{e^{iK\tilde{r}}}{4\pi\tilde{r}} \tag{64}$$

for the transpose of the Helmholtz Green form satisfying radiation boundary conditions for a homogeneous medium with diagonalizable permeability tensor. Note that the matrix exponential can be computed in closed form from the Jordan normal form of K.

If $\epsilon^T {\mu^T}^{-1}$ is diagonal, then $\tilde{g}$ is also diagonal. For symmetric or hermitian ϵ_{ij} and μ_{ij}, this is equivalent to the simultaneous diagonalizability of ϵ_{ij} and μ_{ij}. An important special case of this situation is a homogeneous, magnetically isotropic medium with diagonalizable permitivity tensor, or a biaxial medium. This case is discussed in the next section.

4.1 Biaxial Media

In this section we treat a magnetically isotropic medium ($\mu_{ij} = \mu_0 \delta_{ij}$) with diagonalizable permitivity tensor. For convenience, in this and following sections we scale G by a factor of μ_0 and g by a factor of μ_0^2. If the coordinates system is transformed so that the permittivity tensor is diagonal with eigenvalues ϵ_i, then $g(\mathbf{k})$ has the diagonal elements

$$g_i = \frac{1}{k^2 - k_{0i}^2} \tag{65}$$

where $k_{0i} = \omega\sqrt{\epsilon_i \mu_0}$ and other elements vanish. In physical space,

$$g_i(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^{ik_{0i}r}}{4\pi r} \tag{66}$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$.

By repeated substitution and rearrangement of (49), the series

$$G = g + g\mathbf{k}\mathbf{k}^T g + g\mathbf{k}\mathbf{k}^T g\mathbf{k}\mathbf{k}^T g \; + \; \cdots \tag{67}$$

is obtained. Matrices factor out to the right and left, leaving a scalar geometric series in $\mathbf{k}^T g\mathbf{k}$,

$$G = g + g\left[1 + \mathbf{k}^T g\mathbf{k} + (\mathbf{k}^T g\mathbf{k})^2 \; + \; \cdots\right] \mathbf{k}\mathbf{k}^T g \tag{68}$$

Summing this series gives

$$G = g + \left(\frac{g}{1 - \mathbf{k}^T g\mathbf{k}}\right) \mathbf{k}\mathbf{k}^T g \tag{69}$$

In the electrically isotropic case, $(1 - \mathbf{k}^T g\mathbf{k})g^{-1}$ reduces to $-k_0^2 I$, where $k_0^2 = \omega^2 \epsilon_0 \mu_0$, so that this expression in physical space reduces to the usual isotropic expression [3, 9] for G in terms of g,

$$G = \left(1 + \frac{1}{k_0^2} \star d \star d\right) g_0 I \tag{70}$$

where the derivatives act on the 2–form part of g. Note also that the series (68) is singular for values of $\mathbf{k}$ that represent allowed plane waves, so that

$$\mathbf{k}^T g\mathbf{k} = 1 \tag{71}$$

is another form of the Fresnel equation (55).

5. CONCLUSION

In order to conveniently represent macroscopic electromagnetic properties of a medium, we have defined anisotropic Hodge star operators in which the permitivity and permeability tensors of the medium are embedded. The use of these operators along with other tools of the calculus of differential forms makes expressions concise and simplifies manipulations. We also place the Green form derivation in the context of the usual formal treatment of Green functions, so that the origin of symmetry and self–adjointness properties becomes clear.

The primary result of this paper is a generalization of the usual relationship between the scalar and tensor Green functions for isotropic media. This relationship becomes an integral equation connecting the Helmholtz Green form to the Green form for the electric field which is valid for arbitrarily anisotropic and inhomogeneous media. The Helmholtz Green form satisfying radiation boundary conditions can be obtained exactly in physical space for a homogeneous medium with diagonalizable permittivity tensor. Due to the general advantages in stability of numerical methods for solving integral equations as opposed to differential equations, this relationship may be useful as a basis for numerical techniques for scattering problems in homogeneous media.

One possible extension of this work is investigation of the Neumann series solution for the Green form. There may be particular types of media or source configurations for which the series can be summed or the general term represented in a simple form.

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APPENDIX. THE HODGE STAR OPERATOR

In this appendix, we extend the usual definition of the Hodge star operator to allow its use in expressing the constitutive relations for nonreciprocal or lossy media. We then obtain an expression for the inverse of the monsymmetric star operator and prove the identity $\lambda \wedge \star\nu =$

$\star^{-1}\lambda \wedge \nu$, where λ and ν are p-forms, for the nonsymmetric case. Finally, we generalize the definition of the Laplace–de Rham or wave operator to employ the nonsymmetric star operator.

The star operator $\star$ is a linear mapping from p–forms to $(n-p)$–forms. In terms of a metric [10],

$$\star\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} = g^{i_1 j_1} \ldots g^{i_p j_p} \varepsilon_{j_1 \ldots j_n} \frac{\sqrt{|g|}}{(n-p)!}\, dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n} \tag{72}$$

where ε is the Levi-Civita tensor, g is the determinant of the metric tensor g_{ij}, n is the dimension of space, and g^{ij} is the inverse metric. In R^3 with a positive definite (symmetric) metric, $\star = \star^{-1}$. Other definitions exist, such as that of Flanders [11] and Bamberg and Sternberg [5],

$$\lambda \wedge \nu = (\star\lambda, \nu)\sigma \tag{73}$$

where ν is a p–form, λ is an $(n-p)$–form, σ is the volume element $\sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n$ and $(\ ,\ )$ denotes the inner product of p–forms determined by g^{ij}. Thirring [10] defines the star operator to be

$$\star\lambda = \lambda \lrcorner \sigma \tag{74}$$

where $\lrcorner$ denotes the metric–dependent interior product on differential forms. These definitions can be shown to be equivalent.

For symmetric, positive definite permittivity and permeability tensors, we define $\star_e$ using (72) with the inverse metric $g^{ij} = \epsilon_{ji}/(\det \epsilon_{ij})$ and $\star_h$ with $g^{ij} = \mu_{ji}/(\det \mu_{ij})$. While a metric must be symmetric, the definition (72) can be modified to accomodate nonsymmetric ϵ_{ij} and μ_{ij}, corresponding to nonreciprocal or lossy media. For a nonsymmetric g_{ij}, we use the modified definition

$$\star\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} = g^{i_1 j_1} \ldots g^{i_p j_p} \varepsilon_{j_1 \ldots j_n} \frac{\sqrt{g}}{(n-p)!}\, dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n} \tag{75}$$

for the star operator. This definition agrees with (72) if all of the eigenvalues of g^{ij} are positive and real. In general, if the usual definition (72) were employed with nonsymmetric g_{ij}, the star operator thus obtained would differ from that defined by (75) by the phase $|g|/g$, and the constitutive relations (2) would not be valid. With this modified definition, the relationships (3) and (4) are obtained for $\star_e$, with similar behavior for $\star_h$.

For nonsymmetric g_{ij}, the star operator is no longer proportional to its inverse, since as shown below the inverse of the star operator must be defined using (75) with g^{ij} replaced by its transpose g^{ji}. We give this transposed star operator the symbol $\tilde{\star}$. The inverse star operators $\star_e{}^{-1}$ and $\star_h{}^{-1}$ are thus defined using (75) with $g^{ij} = \epsilon_{ij}/(\det \epsilon_{ij})$ and $\star_h$ with $g^{ij} = \mu_{ij}/(\det \mu_{ij})$.

We prove that $\tilde{\star}$ is proportional to $\star^{-1}$ for a nonsymmetric g^{ij}. Applying the definition (75) and using the shorthand notation $dx^{i_1} \wedge \cdots \wedge dx^{i_p} = dx^{i_1 \ldots i_p}$,

$$\begin{aligned}
\tilde{\star} \star dx^{i_1 \ldots i_p} &= \frac{g}{p!(n-p)!} g^{k_{p+1} j_{p+1}} \cdots g^{k_n j_n} g^{i_1 j_1} \cdots g^{i_p j_p} \varepsilon_{k_{p+1} \ldots k_n k_1 \ldots k_p} \varepsilon_{j_1 \ldots j_n} \, dx^{k_1 \ldots k_p} \\
&= \frac{g}{p!(n-p)!} g^{k_{p+1} j_{p+1}} \cdots g^{k_n j_n} g_{j_{p+1} l_{p+1}} \cdots g_{j_n l_n} \varepsilon^{i_1 \ldots i_p l_{p+1} \ldots l_n}_{k_{p+1} \ldots k_n k_1 \ldots k_p} \, dx^{k_1 \ldots k_p} \\
&= \frac{g}{p!(n-p)!} \delta^{k_{p+1} \ldots k_n}_{l_{p+1} \ldots l_n} (-1)^{p(n-p)} \varepsilon^{i_1 \ldots i_p l_{p+1} \ldots l_n}_{k_1 \ldots k_n} \, dx^{k_1 \ldots k_p} \\
&= \frac{g}{p!(n-p)!} (-1)^{p(n-p)} \frac{(n-p)!}{g} \delta^{i_1 \ldots i_p}_{k_1 \ldots k_p} \, dx^{k_1 \ldots k_p} \\
&= (-1)^{p(n-p)} \, dx^{i_1 \ldots i_p}.
\end{aligned}$$

By linearity the proof extends to general p–forms. Thus,

$$\star^{-1} = (-1)^{p(n-p)} \tilde{\star} \tag{76}$$

so that in R^3, $\star^{-1} = \tilde{\star}$.

We also require the identity $\nu \wedge \star\lambda = \star^{-1}\nu \wedge \lambda$ for p-forms ν and λ. Thirring [10] proves the result for a symmetric star operator. We generalize to the nonsymmetric case. The proof is given for simple forms and extends to the general case by linearity. Applying the definition (75) of the star operator,

$$\begin{aligned}
dx^{i_1 \ldots i_p} \wedge \star dx^{j_1 \ldots j_p} &= \frac{\sqrt{g}}{(n-p)!} g^{j_1 k_1} \cdots g^{j_p k_p} \varepsilon_{k_1 \ldots k_n} \, dx^{i_1 \ldots i_p k_{p+1} \ldots k_n} \\
&= \frac{\sqrt{g}}{(n-p)!} g^{j_1 k_1} \cdots g^{j_p k_p} \varepsilon_{k_1 \ldots k_n} \varepsilon^{i_1 \ldots i_p k_{p+1} \ldots k_n} \, dx^{1 \ldots n} \\
&= \frac{\sqrt{g}}{(n-p)!} g^{j_1 k_1} \cdots g^{j_p k_p} \delta^{i_1 \ldots i_p}_{k_1 \ldots k_p} \, dx^{1 \ldots n}.
\end{aligned} \tag{77}$$

where δ here denotes the permutation tensor. Using an explicit representation [12] for δ, Eq. (77) becomes

$$dx^{i_1 \ldots i_p} \wedge \star dx^{j_1 \ldots j_p} = \frac{\sqrt{g}}{(n-p)!} \sum_{\pi} g^{j_1 i_{\pi(1)}} \cdots g^{j_p i_{\pi(p)}} \mathrm{sgn}(\pi) \, dx^{1 \ldots n}. \tag{78}$$

Rearranging the order of the $g^{j_k i_{\pi(k)}}$, this is transformed into

$$dx^{i_1 \ldots i_p} \wedge \star dx^{j_1 \ldots j_p} = \frac{\sqrt{g}}{(n-p)!} \sum_{\pi} g^{j_{\pi(1)} i_1} \cdots g^{j_{\pi(p)} i_p} \mathrm{sgn}(\pi) \, dx^{1 \ldots n}.$$

Reversing the steps leading to Eq. (78), we find that

$$\begin{aligned}
dx^{i_1 \ldots i_p} \wedge \star dx^{j_1 \ldots j_p} &= \frac{\sqrt{g}}{(n-p)!} g^{k_1 i_1 \ldots k_p i_p} \varepsilon_{k_1 \ldots k_n} \, dx^{j_1 \ldots j_p k_{p+1} \ldots k_n} \\
&= \frac{\sqrt{g}}{(n-p)!} g^{k_1 i_1 \ldots k_p i_p} \varepsilon_{k_1 \ldots k_n} (-1)^{p(n-p)} \, dx^{k_{p+1} \ldots k_n j_1 \ldots j_p}.
\end{aligned}$$

Using the definition of $\tilde{\star}$ and (76) shows that

$$dx^{i_1 \dots i_p} \wedge \star\, dx^{j_1 \dots j_p} \;=\; \star^{-1} dx^{i_1 \dots i_p} \wedge dx^{j_1 \dots j_p}. \tag{79}$$

In R^3, the inverse star operator can be replaced with $\tilde{\star}$.

Finally, we extend the definition of the Laplace–de Rham or wave operator Δ to allow use of the nonsymmetric star operator. Δ is a generalization of the Laplacian. Variation in sign conventions for Δ exists the literature; the two alternatives are found in Bamberg and Sternberg [5] and Thirring [10]. We choose Thirring's definition, since it agrees with the sign of the usual vector Laplacian. Accordingly, we define

$$\Delta\alpha = (-1)^{n(p+1)} \left[(-1)^n \star d\tilde{\star}d + d\tilde{\star}d\star\right] \alpha \tag{80}$$

where α is a p-form. This is equivalent to Thirring's definition for a positive definite (symmetric) metric. For a nonpositive definite metric with real eigenvalues it differs by the sign $|g|/g = (-1)^s$, where s is the signature of g_{ij}. For a constant g^{ij}, in a particular coordinate system (80) reduces to

$$\Delta(\omega\, dx^{i_1 \dots i_p}) \;=\; g^{ij} \frac{\partial^2 \omega}{\partial x_i \partial x_j}\, dx^{i_1 \dots i_p} \tag{81}$$

which is the usual expression for the Laplacian if g^{ij} is equal to the euclidean metric δ_{ij}.

References

[1] D. N. Moskvin, V. P. Romanov, and A. Y. Val'kov, "Green's function of the electromagnetic field in biaxial media," *Phys. Rev. E*, vol. 48, no. 2, pp. 1436–1446, 1993.

[2] W. Ren, "Contributions to the electromagnetic wave theory of bounded homogenous anisotropic media," *Phys. Rev. E*, vol. 47, pp. 664–673, Jan. 1993.

[3] K. F. Warnick and D. V. Arnold, "Electromagnetic green functions using differential forms," *Journal of Electromagnetic Waves and Applications*, vol. 10, no. 3, pp. 427–438, 1996.

[4] G. de Rham, *Differentiable Manifolds.* New York: Springer-Verlag, 1984.

[5] P. Bamberg and S. Sternberg, *A Course in Mathematics for Students of Physics*, vol. II. Cambridge: Cambridge University Press, 1988.

[6] W. C. Chew, *Waves and Fields in Inhomogenous Media.* New York: IEEE Press, 1995.

[7] P. Boulanger and M. Hayes, "Electromagnetic planes waves in anisotropic media: An approach using bivectors," *Phil. Trans. R. Soc. Lond. A*, vol. 330, pp. 335–393, 1990.

[8] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves.* Piscataway, NJ: IEEE Press, 1994.

[9] J. A. Kong, *Electromagnetic Wave Theory.* New York: John Wiley & Sons, 1990.

[10] W. Thirring, *Classical Field Theory*, vol. II. New York: Springer-Verlag, 1978.

[11] H. Flanders, *Differential Forms with Applications to the Physical Sciences.* New York, New York: Dover, 1963.

[12] S. Hassani, *Foundations of Mathematical Physics.* Boston: Allyn and Bacon, 1991.