Differential Forms in Electromagnetic Field Theory

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I. INTRODUCTION

The calculus of differential forms has been applied to electromagnetic field theory in several papers and texts, some of which are cited in the references. Despite this work, differential forms are underused in applied electromagnetics research, partly due to the abstract viewpoint used in most treatments. Differential forms have a simple and intuitive geometrical interpretation [1], [2] which allows problems to be attacked intuitively and sometimes solved before the formal details are developed. It has been known for decades that forms allow Maxwell's laws to be written elegantly; we find that this is true of other expressions as well. Compared to the usual vector formulation, derivations are often cleaner and final results easier to apply, making the calculus of differential forms a natural tool for applied electromagnetics.

The calculus of differential forms can be presented as concretely as vector analysis. Sec. II is an abbreviated version of such a treatment. We give Maxwell's laws and their geometrical interpretation and show how Ampere's law and the curl operator become as intuitive as Gauss's law and the divergence. Boundary conditions are quite simple when written using vectors, but there is no obvious graphical interpretation. In Sec. III, we show that differential forms make boundary conditions geometrically obvious. This kind of insight, applied to more complex problems, can guide one to a solution and help interpret the solution when it is discovered.

In Sec. IV we redevelop the dyadic Green function as a double differential form. The usual results are simplified by the absence of surface normal vectors. Sec. V extends the Green forms to anisotropic media. Differential forms are natural for these applications. Compared to the dyadic formulation, final results are simpler and derivations more algebraic.

Differential forms do not replace vectors; ideally forms and vectors are used interchangeably as appropriate to a particular problem. Forms and vectors together are the subset of tensor analysis that optimally combines generality and concreteness. We hope to show in this paper that differential forms make Maxwell's laws and some of their basic applications more intuitive and are a natural and powerful research tool in applied electromagnetics.

II. THE CALCULUS OF DIFFERENTIAL FORMS

The calculus of differential forms is the calculus of quantities that can be integrated. The degree of a form is the dimension of the region over which it is integrated, so that in R^3 there are 0-forms, 1-forms, 2-forms and 3-forms.

A. Differential Forms; Exterior Product

1-forms are integrated over paths, and are represented graphically by surfaces, as in Fig. 1a. The surfaces of a 1-form have an associated orientation, the direction of the vector dual to the form. The general 1-form is $a\,dx + b\,dy + c\,dz$, with dual vector $a\hat{x} + b\hat{y} + c\hat{z}$ in the euclidean metric. The integral of a 1-form over a path is the number of surfaces pierced by the path, taking into account the orientation of the surfaces and the direction of integration.

2-forms are integrated over surfaces. A basis for the 2-forms is $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$, with dual vectors $\hat{x}$, $\hat{y}$, $\hat{z}$. The wedge represents the exterior product, which is anticommutative, so that $dx \wedge dy = -dy \wedge dx$ and $dx \wedge dx = 0$. Wedges are often dropped for compactness.

Graphically, 2-forms are tubes (Fig. 1b). As the coefficients of a 2-form increase, the tubes become narrower and more dense. The tubes are oriented in the direction of the associated dual vector. The integral of a 2-form over a surface is the number of tubes passing through the surface, with respect to the relative orientations of the tubes and the surface.

A 3-form is a volume element, represented by boxes (Fig. 1c). The greater the magnitude of a 3-form's coefficient, the smaller and more closely spaced are the boxes. The integral of a 3-form over a volume is the number of boxes inside the volume, where each box is weighted by the sign of the 3-form. A 3-form is dual to its coefficient. Finally, a 0-form is a function. Forms of degree greater than three vanish by the anticommutativity of the exterior product.

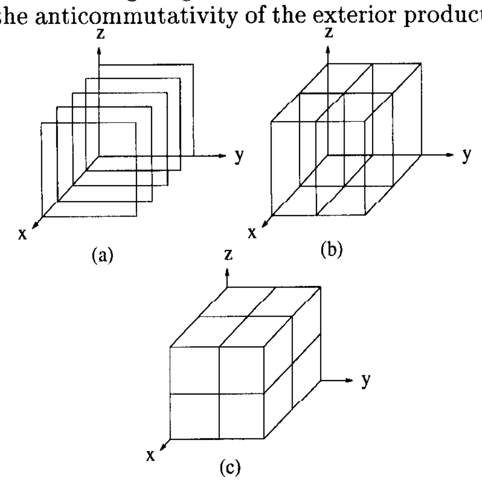

Fig. 1. (a) The 1-form dx. (b) The 2-form $dy\,dz$. Tubes in the z direction are formed by the superposition of the surfaces of dy and the surfaces of dz. (c) The 3-form $dx\,dy\,dz$, with three sets of surfaces that create boxes.

The electric and magnetic field intensities E and H are 1-forms; their surfaces represent equipotentials. The electric and magnetic flux densities D and B are 2-forms, as well as the electric current density J. The electric charge density, ρ, is a 3-form, with each box representing a small amount of charge. These forms are dual to the usual vector and scalar quantities.

B. The Interior Product

The interior product of a vector and a form is the usual tensor contraction. Using a metric, the interior product of forms can be defined. In the euclidean metric, $dx \lrcorner\, dx = dy \lrcorner\, dy = dz \lrcorner\, dz = 1$ and all other combinations vanish. For 1-forms and 2-forms,

$$dz \lrcorner (\, dz \wedge dx) = -\, dy \lrcorner (\, dx \wedge dy) = dx$$
$$dx \lrcorner (\, dx \wedge dy) = -\, dz \lrcorner (\, dy \wedge dz) = dy$$
$$dy \lrcorner (\, dy \wedge dz) = -\, dx \lrcorner (\, dz \wedge dx) = dz$$

and $dx \lrcorner (\, dy \wedge dz) = dy \lrcorner (\, dz \wedge dx) = dz \lrcorner (\, dx \wedge dy) = 0$. Graphically, the interior product removes the surfaces of the first form from those of the second, as will be seen in Sec. III.

C. The Exterior Derivative; Maxwell's Laws

The exterior derivative can be written formally as

$$d = (\frac{\partial}{\partial x}\, dx + \frac{\partial}{\partial y}\, dy + \frac{\partial}{\partial z}\, dz)\wedge \qquad (1)$$

and acts like the vector gradient operator on 0-forms, the curl on 1-forms and divergence on 2-forms. The exterior derivative of $f\, dx$, for example, is $f_y\, dy\, dx - f_z\, dz\, dx$, where subscripts represent partial derivatives.

The generalized Stokes theorem is

$$\int_M d\omega = \int_{\partial M} \omega \qquad (2)$$

where ω is a p-form and ∂M is the boundary of a $(p+1)$-dimensional region M. This relationship contains the fundamental theorem of calculus, the vector Stokes theorem and the divergence theorem as special cases.

Using the exterior derivative, Maxwell's laws can be written

$$dE = -\frac{\partial}{\partial t} B \qquad dD = \rho$$
$$dH = \frac{\partial}{\partial t} D + J \qquad dB = 0.$$

By Stokes theorem, Gauss's law for the electric field has the geometrical interpretation that boxes of charge produce tubes of electric flux, as shown in Fig. 2a. Ampere's law shows [1] that in a similar way tubes of electric current or time–varying electric flux produce magnetic field intensity surfaces (Fig. 2b). With vectors, Ampere's law and the curl operator are not as intuitive as Gauss's law and the divergence. With differential forms, Ampere's law obtains a geometrical meaning that is as simple as (and clearly related to) that for Gauss's law.

D. The Star Operator; Constitutive Relations

The Hodge star operator is a set of isomorphisms between p-forms and $(n-p)$-forms, where n is the dimension of the underlying space. In R^3 with the euclidean metric,

$$\star\, dx = dy\, dz, \quad \star\, dy = dz\, dx, \quad \star\, dz = dx\, dy$$

and $\star 1 = dx\, dy\, dz$. Also, $\star\star = 1$, so that the star operator is its own inverse. The interior product can be written in terms of the star operator, so that $a \lrcorner b = \star(\star b \wedge a)$.

The constitutive relations in free space are $D = \epsilon_0 \star E$ and $B = \mu_0 \star H$. Graphically, tubes of flux are perpendicular to surfaces of field intensity, as in Fig. 2c.

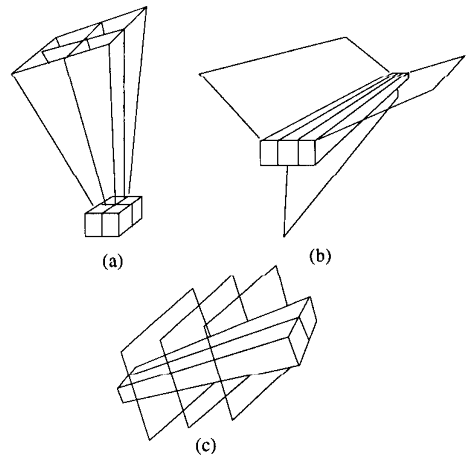

Fig. 2. (a) Gauss's law: boxes of electric charge produce tubes of electric flux. (b) Ampere's law: tubes of current produce magnetic field surfaces. (c) Tubes of D are perpendicular to surfaces of E, since $D = \epsilon_0 \star E$.

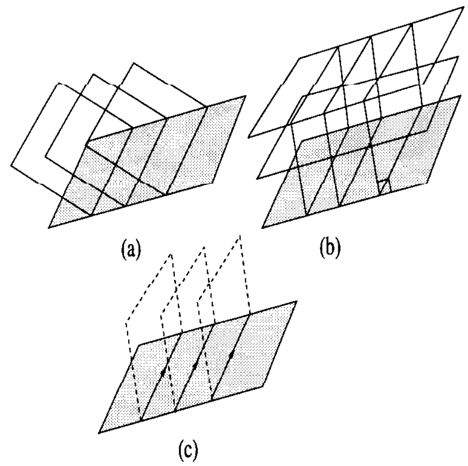

Fig. 3. (a) The field discontinuity $H_1 - H_2$, which has the same intersection with the boundary as J_s. (b) The exterior product $n \wedge [H_1 - H_2]$ yields tubes running along the boundary, with sides perpendicular to the boundary. (c) The interior product with n removes the surfaces parallel to the boundary, leaving surfaces that intersect the boundary along the lines representing the 1-form J_s.

III. BOUNDARY CONDITIONS

Boundary conditions on the electromagnetic field can be written using the operator $n \lrcorner n \wedge$, where n is the normalized exterior derivative of a function f that vanishes along a boundary surface. As proved in [3],

$$\begin{aligned} n \lrcorner (n \wedge [E_1 - E_2]) &= 0 \\ n \lrcorner (n \wedge [H_1 - H_2]) &= J_s \\ n \lrcorner (n \wedge [D_1 - D_2]) &= \rho_s \\ n \lrcorner (n \wedge [B_1 - B_2]) &= 0 \end{aligned}$$

where subscripts represent values above ($f > 0$) and below ($f < 0$) the boundary, J_s is the surface current density 1-form and ρ_s is the surface charge density 2-form.

These expressions for boundary conditions have a simple geometric interpretation. The discontinuity $H_1 - H_2$, for example, is a 1-form with surfaces that intersect the boundary along the lines of the 1-form J_s (Fig. 3a). From the fields above and below a boundary one knows immediately what sources lie on the boundary.

In the expression for J_s, the exterior product $n \wedge (H_1 - H_2)$ creates tubes with sides perpendicular to the boundary (Fig. 3b). The interior product $n \lrcorner (n \wedge [H_1 - H_2])$ removes the surfaces that were added by the exterior product, as shown in Fig. 3c. The total effect of the operator $n \lrcorner n \wedge$ is to select the component of $H_1 - H_2$ with surfaces perpendicular to the boundary.

Unlike other differential forms of electromagnetics, J_s is not dual to the usual surface current density vector $\mathbf{J_S}$. The expression for current through a path P is

$$I = \int_P \mathbf{J_S} \cdot (\hat{n} \times d\hat{s}) \tag{3}$$

where $\hat{n}$ is a surface normal and $\hat{s}$ is tangent to the path. Using the 1-form J_s, this simplifies to

$$I = \int_P J_s. \tag{4}$$

Vector integrals such as (3) are complicated by the unwieldy behavior of vectors under change of variables. Like all integrals of differential forms, Eq. (4) can be evaluated by pullback. Each occurrence of x, y and z is replaced with $x(s)$, $y(s)$ and $z(s)$, where s is a parameter for the path P. Jacobian factors enter automatically.

IV. GREEN FORMS

The dyadic Green function can be represented by a double form [4], which is a sum of tensor products of forms. The $1 \otimes 1$ Green form G is given by

$$G = (1 + \frac{1}{k^2} d \star d \star) g I \tag{5}$$

where I is the unit double form $I = dx\, dx' + dy\, dy' + dz\, dz'$ and g is the usual scalar Green function $e^{ikr}/4\pi r$. Primes denote source coordinates.

As shown in [5], for time-harmonic sources in a volume V' the observed electric field is

$$E = i\omega\mu \int_{V'} G\wedge J' + \int_{\partial V'} (\star dG\wedge E' + i\omega\mu G\wedge H'). \quad (6)$$

Note that there are no surface normal vectors in this expression. For a surface current on the boundary of a surface S,

$$E = i\omega\mu \int_S G \wedge J'_s$$

where J'_s is the surface current 1-form defined in the previous section.

V. GREEN FORMS FOR ANISOTROPIC MEDIA

In this section, we generalize the Green forms to an anisotropic, non-bianistropic and reciprocal medium. The permittivity and permeability tensors characterizing a medium can can be considered as metrics and embedded in the Hodge star operator, so that the constitutive relations become $D = \star_e E$ and $B = \star_h H$.

The anisotropic Green $2 \otimes 1$ form satisfies

$$(-\star_h d\star_h d + \omega^2 \star_h \star_e) G(\mathbf{r}_2, \mathbf{r}_1) = -\delta(\mathbf{r}_2, \mathbf{r}_1) I \quad (7)$$

where I is the unit $2 \otimes 1$ form and operators act on the 1-form ($\mathbf{r}_1$) part of G. In terms of sources and external fields, the electric field in observation space is

$$E = i\omega\star_h \int_{V'} G\wedge J' + \star_h \int_{\partial V'} (\star_h dG\wedge E' + i\omega G\wedge H') \quad (8)$$

where as in the previous section primes denote source ($\mathbf{r}_1$) coordinates. The surface integral terms generalize Huygens' principle to anisotropic media. This expression is nearly identical to the isotropic result, Eq. (6).

We define a generalization of the scalar Green function, the $2 \otimes 1$ form g that satisfies

$$(\Delta_h + \omega^2 \star_h \star_e) g = -\delta I \quad (9)$$

where the (magnetic) wave operator Δ_h is defined by $(-1)^{p+1} d\star_h d\star_h \alpha + (-1)^p \star_h d\star_h d\alpha$ for a p-form α. In an isotropic medium, $g = \mu^2 g_0 I$, where g_0 is the usual scalar Green function. For a biaxial medium, g has diagonal elements

$$g_{ii} = \mu^2 e^{i\omega\sqrt{\epsilon_i \mu}} (4\pi r)^{-1} \quad (10)$$

where the ϵ_i are the diagonal elements of the permittivity tensor $\bar{\bar{\epsilon}}$.

From Eq. (9) it can be shown that G and g are related by

$$G = g - \int g \wedge \star_h d\star_h dG \quad (11)$$

where derivatives act on the 2-form part of G. This is a Dyson-like integral equation with the derivative $\star_h d\star_h d$ in place of a perturbing potential. By analogy with quantum field theory, g plays the role of a free propagator.

Specialized to particular sources and fields, Eq. (11) should be solvable using well-known numerical methods. By repeated substitution a Neumann series solution for G can also be obtained.

Compared to the dyadic formulation, the derivation of these results is simplified since the anisotropic star operators have the same algebraic properties as the euclidean star operator. For this application, the calculus of differential forms is ideal.

VI. CONCLUSION

Differential forms make Maxwell's laws more intuitive, boundary conditions more geometrical and Green functions easier to work with. The simplification found in these areas will likely extend to other problems. Since the calculus of forms can be introduced in the same simple, physical manner as vector analysis, with important advantages in providing physical insight, a combination of differential forms and vectors could benefit teaching and research in electromagnetics.

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(For a complete bibliography, send email or visit the website given above.)