Electromagnetic Boundary Conditions and Differential Forms

Karl F. Warnick, Richard H. Selfridge and David V. Arnold

Abstract—We develop a new representation for electromagnetic boundary conditions involving a boundary projection operator defined using the interior and exterior products of the calculus of differential forms. This operator expresses boundary conditions for fields represented by differential forms of arbitrary degree. With vector analysis, the field intensity boundary conditions require the cross product, whereas the flux boundary conditions use the inner product. With differential forms, the field intensity and flux density boundary conditions are expressed using a single operator. This boundary projection operator is readily applied in practice, so that our work extends the utility of the calculus of differential forms in applied electromagnetics.

1. INTRODUCTION

In this paper we derive a new formulation for the boundary conditions at a discontinuity in the electromagnetic field using differential forms. The utility of the calculus of differential forms in electromagnetic field (EM) theory has been demonstrated by Deschamps [1], Baldomir [2], Schleifer [3], Thirring [4], Burke [5, 6] and others. The intent of this paper is to extend the range of engineering problems for which differential forms are useful by providing a practical means of working with boundary conditions.

Thirring [4] and Burke [5, 6] treat boundary conditions using the calculus of differential forms. Thirring's approach is similar to ours, but we extend his methods by introducing a boundary projection operator to express boundary conditions for forms of arbitrary degree in a space of arbitrary dimension. The expressions for junction conditions on field intensity and flux density, for example, are identical in form.

For many electromagnetic quantities, the vector representation and the representation as a differential form are duals, so that their components differ only by metrical coefficients. This is not the case for the surface current and charge density twisted forms yielded by the boundary projection operator. It is simpler to compute, for example, total current through a path using the surface current twisted 1-form than using the usual surface current vector.

In Sec. 2 we derive an expression for boundary sources at a field discontinuity using the boundary projection operator. In Sec. 3 we provide a simple computational example to illustrate the method. Due to the unfamilarity of most engineers with differential forms and associated notation, our treatment is more elementary than is usual. Accordingly, we also provide an introduction to the interior product and twisted forms in an Appendix. This work shows that the calculus of differential forms is useful for practical EM problems involving boundary conditions.

2. BOUNDARY CONDITIONS

In this section we derive an expression for sources on a boundary where the electromagnetic field is discontinuous. We find that generalized boundary conditions can be given using a boundary projection operator. We then discuss the resulting boundary conditions for magnetic field intensity and electric flux density.

2.1 Representing Surfaces With 1-forms

In an n-dimensional space, a 1-form is represented graphically by (n-1)-dimensional hypersurfaces. In n-space, a boundary is also an (n-1)-dimensional hypersurface. Thus, we can use 1-forms to represent boundaries. If a continuous function $f(x_1, ..., x_n)$ vanishes (or is constant) along a boundary, then the 1-form df graphically has a surface that lies on the boundary. The surface of a paraboloid reflector antenna, for example, is given by $-x^2 - y^2 + az = 0$, so that the unnormalized boundary 1-form is $-2x\,dx - 2y\,dy + a\,dz$. A randomly rough surface is written $-h(x, y) + z = 0$, giving the boundary 1-form $-dh + dz$.

For the remainder of this paper we will use the notation

$$n \equiv \frac{df}{|df|} = \frac{df}{\sqrt{df \lrcorner df}} \tag{1}$$

where $\lrcorner$ is the interior product, which is discussed in the Appendix. The 1-form n is dual to the usual surface normal vector $\hat{n}$.

2.2 Derivation of the Boundary Projection Operator

In three dimensions, let E be the electric field intensity 1-form, H the magnetic field intensity 1-form, D the electric flux density 2-form, B the magnetic flux density 2-form, J the electric current density 2-form and ρ the charge density 3-form. Then Maxwell's laws are

$$\begin{aligned} dE &= -\frac{\partial}{\partial t}B \\ dH &= \frac{\partial}{\partial t}D + J \\ dD &= \rho \\ dB &= 0. \end{aligned} \tag{2}$$

Each equation equates the exterior derivative of a differential form to the sum of a source and a nonsingular field term, of which one or both may vanish. Recognizing that Maxwell's laws have a common form, we can derive an expression for the boundary sources that is the same for both field intensity (1-forms) and flux density (2-forms).

Let α be a p-form with $p < n$ (where n is the dimension of space) that represents a field with a $(p+1)$-form β as a source, so that

$$d\alpha = \gamma + \beta \tag{3}$$

where γ is nonsingular. Let $f = 0$ represent a boundary, where f is C^1 and vanishes only along the boundary. Let α equal α_2 for $f > 0$ and α_1 for $f < 0$.

We can write $\alpha = (\alpha_2 - \alpha_1)\theta(f) + \alpha_1$, where θ is the unit step function. Then

$$\begin{aligned} \gamma + \beta &= d\{(\alpha_2 - \alpha_1)\theta(f) + \alpha_1\} \\ &= \delta(f) df \wedge (\alpha_2 - \alpha_1) + \theta(f) d(\alpha_2 - \alpha_1) + d\alpha_1. \\ &= \tilde{\delta}(f) n \wedge (\alpha_2 - \alpha_1) + \theta(f) d(\alpha_2 - \alpha_1) + d\alpha_1 \end{aligned} \tag{4}$$

where δ is the Dirac delta function and $\tilde{\delta}(f)$ is $\delta(x^1 - x_0^1) \cdots \delta(x^n - x_0^n)$ such that the point $(x_0^1, ..., x_0^n)$ lies on the boundary and $\delta(f) = \frac{\tilde{\delta}(f)}{\sqrt{df \lrcorner df}}$. The singular parts of both sides of (4) must be equal, so that

$$\beta' = \tilde{\delta}(f) n \wedge (\alpha_2 - \alpha_1) \tag{5}$$

where β' is the singular part of β, representing the boundary source along $f = 0$. Since the source β' is confined to the boundary, it can be written [4]

$$\beta' = \tilde{\delta}(f) n \wedge \beta_s \tag{6}$$

where β_s is a p-form, the restriction of β' to the boundary. Integrating (5) and (6) over a region containing the boundary shows that the equality

$$n \wedge \beta_s = n \wedge (\alpha_2 - \alpha_1). \tag{7}$$

must hold on the boundary. We then take the interior product of both sides of (7) by n and apply the identity (30) to obtain

$$\begin{aligned} n \lrcorner (n \wedge (\alpha_2 - \alpha_1)) &= n \lrcorner (n \wedge \beta_s) \\ &= (n \lrcorner n) \wedge \beta_s - n \wedge n \lrcorner \beta_s. \end{aligned} \tag{8}$$

Because n is normalized, $n \lrcorner n = 1$. Since β_s is by definition confined to the boundary, $n \lrcorner \beta_s = 0$. Graphically, the surfaces of β_s are perpendicular to the boundary because β_s can contain no factor proportional to n. Applying $n \lrcorner \beta_s = 0$ to (8), we have

$$\beta_s = n \lrcorner (n \wedge (\alpha_2 - \alpha_1)) \tag{9}$$

which is the central result of this paper.

Eq. (9) applies to both nontwisted and twisted forms. As noted below, electromagnetic sources are conveniently represented by twisted forms. If β_s is a twisted form, we must provide an outer orientation (see Appendix A and Burke [6]) for β_s. The orientation is given by

$$\{(\beta_s, \Omega_s)\} \wedge n \wedge \beta_s = \Omega \tag{10}$$

where $\{(\beta_s, \Omega_s)\}$ is a nontwisted $(n-p-1)$-form specifying the outer orientation of β_s and Ω is a volume element (n-form) serving as an orientation for the surrounding n-space. Ω_s is a volume element in the boundary, but need not be found in order to obtain the outer orientation of β_s. Given any boundary, there are two possible choices for n. The orientation specified by (10) is easily seen to be independent of that choice. In right-handed coordinates, an equivalence can be made between inner and outer orientations, and the orientation of β_s can be taken as that of the $(p+1)$-form $n \wedge \beta_s$.

In Sections 2.4 and 2.5, where (9) is specialized to surface current and surface charge densities, we provide a simpler means for orienting surface sources. We find that the need for an orientation for β_s corresponds precisely to conventions used with the vector calculus when integrating $\mathbf{J}_s$ and the scalar surface charge density q_s.

Since (9) applies to any situation for which a law of the form (3) is valid, we can use Maxwell's laws (2) to write at a boundary $f = 0$,

$$\begin{aligned} n \lrcorner (n \wedge (E_2 - E_1)) &= 0 \\ n \lrcorner (n \wedge (H_2 - H_1)) &= J_s \\ n \lrcorner (n \wedge (D_2 - D_1)) &= \rho_s \\ n \lrcorner (n \wedge (B_2 - B_1)) &= 0 \end{aligned} \tag{11}$$

where J_s is the surface current twisted 1-form and ρ_s is the surface charge twisted 2-form. In four-space we have $dF = 0$ and $d \star F = j$, where $F = B + E \wedge dt$, $\star F = D - H \wedge dt$ and

$j = \rho - J \wedge dt$. We can express all four boundary conditions as

$$\begin{aligned} n \lrcorner (n \wedge (F_2 - F_1)) &= 0 \\ n \lrcorner (n \wedge (\star F_2 - \star F_1)) &= j_s \end{aligned} \tag{12}$$

where $j_s = \rho_s - J_s \wedge dt$ and units are suitably normalized.

The operator $n \lrcorner n \wedge$ might be termed the *boundary projection operator*. Graphically, this operator removes the component of a form with surfaces parallel to the boundary. The boundary projection of a 1-form has surfaces perpendicular to the boundary. The boundary projection of a 2-form has tubes perpendicular to the surface at every point.

2.3 Decomposition of Forms at a Boundary

Any form ω of degree less than the dimension of the space we are working in can be decomposed using the boundary projection operator and its complementary operation $\star^{-1} n \lrcorner n \wedge \star$, so that

$$\omega = n \lrcorner (n \wedge \omega) + \star^{-1} [n \lrcorner (n \wedge \star \omega)] \tag{13}$$

where the second term on the right-hand side is the component of ω with surfaces parallel to the boundary. This identity can be proved by expressing ω in terms of orthonormal 1-forms dx^i at a point, verifying (13) for $n = dx^i$ and extending to the general case $n = n_i \, dx^i$, where $\sum_i (n_i)^2 = 1$ using the linearity of the exterior and interior products. The operator $\star^{-1} \, n \lrcorner n \wedge \star$ can be used to obtain the arbitrary part of a field at a boundary.

We discuss special cases of the boundary projection operator for 1-forms and 2-forms below.

2.4 Surface Current

Surface current is represented by a twisted 1-form. Fig. 1 shows how this 1-form is obtained from the magnetic field intensity at a boundary (Fig. 1a). Fig. 1b shows the 1-form $(H_2 - H_1)$. (Note that the 1-forms H_2 and H_1 are both defined above and below the boundary even though each represents the field on one side of the boundary only. Thus, we can consider

$H_2 - H_1$ as being defined for all space, even though only its value on the boundary is of interest.) Fig. 1c shows $n\rfloor(n \wedge (H_2 - H_1))$. Fig. 1d is the restriction of this 1-form to the boundary, along with the corresponding vector $\mathbf{J}_\mathrm{s}$.

Graphically, the boundary projection operator removes any component of $H_2 - H_1$ with surfaces parallel to the boundary. Physically, this expresses both the requirement that surface current can only flow along the boundary and the arbitrariness of the normal component of $H_2 - H_1$ at the boundary.

Most forms in EM theory, such as E, H, D and B, are dual to the corresponding vector quantity, so that components differ only by metrical coefficients. The twisted surface current 1-form J_s, however, is not the dual of the vector $\mathbf{J}_\mathrm{s}$. The nontwisted 1-form with the same components as $\mathbf{J}_\mathrm{s}$ does not satisfy the simple definition given below in (14).

The surface current 1-form J_s can be defined in terms of the flow vector $\mathbf{v}$ of a surface charge distribution ρ_s. The current density 2-form J is $J = \mathbf{v}\rfloor\rho$. The surface current density is $J_s = -\mathbf{v}\rfloor\rho_s$. If $\rho_s = q\,dx\,dy$ and the flow field $\mathbf{v} = v\hat{y}$, then $J_s = -v\hat{y}\rfloor q\,dx\,dy = qv\,dx$. The 1-form dual to $\mathbf{J}_s$ would be $qv\,dy$.

The integral of the surface current density over a path should yield the total current through the path. The 1-form J_s as obtained using the boundary projection operator satisfies this definition:

$$I = \int_P J_s \tag{14}$$

where P is a path. The sense of I is with respect to the direction of the 2-form $n \wedge s$, where s is the 1-form dual to the tangent vector $\mathbf{s}$ of P (so that s is a 1-form with surfaces perpendicular to the path P and oriented in the direction of integration).

The simple integral in (14) replaces a much more cumbersome vector expression. The total current through P using the usual surface current vector is

$$I = \int_P \mathbf{J}_\mathrm{s} \cdot (\hat{n} \times d\mathbf{s}) \tag{15}$$

where the sense of I is relative to the direction of $\hat{n} \times d\mathbf{s}$. Note that this is the same as the reference direction $n \wedge s$ of (14).

The integral in (14) is evaluated in practice by the method of pullback. If a surface S is parameterized by $x = \alpha_1(u,v)$, $y = \alpha_2(u,v)$, $z = \alpha_3(u,v)$ where u and v range over some subset T of the u–v plane, then the integral of $\omega = \omega_1(x,y,z)\,dx+\omega_2(x,y,z)\,dy+\omega_3(x,y,z)\,dz$ over S is

$$\begin{aligned}\int_S \omega &= \int_T \alpha^*\omega \\ &= \int_T \omega_1(\alpha_1,\alpha_2,\alpha_3)d\alpha_1+\omega_2(\alpha_1,\alpha_2,\alpha_3)d\alpha_2+\omega_3(\alpha_1,\alpha_2,\alpha_3)d\alpha_3 \end{aligned} \tag{16}$$

where the superscript $*$ denotes the pullback operation. After pullback, the integrand becomes a 1-form in du and dv. Partial derivatives of the coordinate transformation enter naturally via the exterior derivatives of α_1, α_2 and α_3. Eq. (16) is easily generalized to integrals of 2-forms.

2.5 Surface Charge

Surface charge density due to discontinuous electric flux density at a boundary is represented by the twisted 2-form $\rho_s = n\lrcorner(n\wedge(D_2-D_1))$. The boundary projection operator removes any component of $D_2 - D_1$ with tubes parallel to the boundary, as shown in Fig. 2.

The 2-form ρ_s obtained using the boundary projection operator differs from the usual value $q_s = \hat{n}\cdot(\mathbf{D_2}-\mathbf{D_1})$ because ρ_s is a 2-form, whereas q_s is a scalar. The total charge on an area A of a boundary with surface charge ρ_s is

$$Q = \int_A \rho_s. \tag{17}$$

Q is positive for positive surface charge if the 2-form ω that satisfies $n\wedge\omega = \Omega$ also satisfies $\int_A \omega > 0$, where Ω is the standard volume element, $dx\,dy\,dz$ in rectangular coordinates. This corresponds exactly to the convention of choosing dS in $Q = \int_A q_s dS$ (where q_s is the usual surface charge density scalar) such that $\int_A dS$ is positive. The sign of the charge represented by ρ_s can also be found by computing $\frac{n\wedge\rho_s}{\Omega}$.

2.6 Comparison to Burke's Pullback Method

Burke [6] derives an expression for boundary sources using the method of pullback. In his notation, boundary conditions have the form $[H] = J_s$ and $[D] = \rho_s$. The square brackets denote the sum of the pullback of H above the boundary to the boundary and the pullback of H below the boundary to the boundary, so that

$$J_s = [H] \equiv \alpha_2^* H_2 + \alpha_1^* H_1 \tag{18}$$

where α_2 and α_1 are functions of the space above and below the boundary into the boundary.

The pullback method has a concise and elegant proof in [6]. By transforming to a coordinate system $x^1, \ldots, x^n$ such that a boundary is given by $x^1 = 0$, it can be shown that Burke's formulation is equivalent to $n \lrcorner n \wedge$ for $n = dx^1$. Although the pullback boundary conditions are mathematically very natural, the boundary projection operator has the advantage that (as with the usual vector formulation) the boundary sources are always expressed in the same coordinates as the fields.

3. EXAMPLE

A conducting boundary lies along the surface $z = \cos y$. Above the boundary the magnetic field is $H_2 = H\,dx$. Below the field is zero. This is shown in Fig. 3a.

We can represent the surface by $f(x, y, z) = z - \cos y = 0$. Computing the normalized exterior derivative of this function,

$$\begin{aligned} n &= \frac{d(-\cos y + z)}{|d(-\cos y + z)|} \\ &= \frac{\sin y\, dy + dz}{\sqrt{1 + \sin^2 y}}. \end{aligned} \tag{19}$$

The boundary projection of H_2 is

$$\begin{aligned} J_s &= n \lrcorner n \wedge H_2 \\ &= \frac{H}{1 + \sin^2 y} (\sin y dy + dz) \lrcorner (\sin y\, dy\, dx + dz\, dx) \\ &= \frac{H}{1 + \sin^2 y} (dx + \sin^2 y\, dx) \\ &= H\, dx. \end{aligned} \tag{20}$$

Fig. 3c shows the 1-form $J_s = H\,dx$ restricted to the boundary. The direction of the arrow along the lines of $H\,dx$ is the orientation of the 2-form $n \wedge J_s$.

Compare the final expression for J_s in (20) to the vector surface current obtained for the same field and boundary, $\mathbf{J_s} = \frac{H}{\sqrt{1+\sin^2 y}}(\hat{y} - \sin y \hat{z})$. The differential form $J_s = H\,dx$ indicates clearly that the total current crossing a path in the boundary is simply the extent of the path in the x direction scaled by the factor H. This is not obvious from the vector expression.

4. CONCLUSION

The boundary projection operator allows one to express electromagnetic boundary conditions using differential forms, using the same operator for both field intensity and flux density. The different appearances of the field intensity and flux density boundary conditions expressed using vector analysis is merely an artifact of the mathematical language. The differential forms for boundary sources obtained via the boundary projection operator differ from the vectors obtained by standard methods. The surface current 1-form J_s, for example, has a more natural definition than the usual surface current vector $\mathbf{J_s}$. This 1-form is readily integrated to yield total current over a path, whereas a vector perpendicular to the path and tangent to the boundary is required to evaluate the integral for total current using the surface current vector. The surface current vector also obscures intuitive properties of the boundary source that are clearly evident when the source is represented by the 1-form J_s. Graphically, a source form is the intersection of the field forms with the boundary, for both the field intensity and flux density cases.

This method for representing boundary conditions can be easily applied in practical problems, and so helps to open the way for the use of differential forms on a regular basis in engineering electromagnetics.

Acknowlegements: This material is based in part upon work supported under a National Science Foundation Graduate Fellowship to KFW.

APPENDIX A. THE INTERIOR PRODUCT; TWISTED FORMS

In this appendix we provide an introduction to the interior product and twisted forms; [6, 7] and others provide more comprehensive treatments of the same topics. The interior product combines a vector (or by abuse of notation, a 1-form) and a p-form to produce a $(p-1)$-form. Twisted forms differ from non-twisted forms in that a twisted form has an outer orientation rather than an inner orientation and so changes sign relative to a nontwisted form under reflection.

A.1 The Interior Product

A differential such as dx is traditionally viewed as an infinitesimal increment of the coordinate x. From the differential geometric point of view, dx is actually a basis element of R^{n*}, the space of linear functions from vectors in R^n into R. A basis for R^{n*} is $dx^1, \ldots, dx^n$, which act on basis vectors $dx_1, \ldots, dx_n$ of R^n (written $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ by the mathematician) as

$$dx^i[\, dx_j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{21}$$

where the square brackets indicate that the differential is a function with the vector inside the brackets as its argument. By linearity, (21) extends to arbitrary 1-forms and vectors.

This definition of 1-forms leads directly to the graphical representation of Misner, *et al* [8]. 1-forms become surfaces in space. The value of a 1-form operated on a vector is the number of surfaces of the 1-form pierced by the vector (see [8] for extensive figures illustrating this point).

Integration can be defined naturally by using the 1-form to be integrated as a linear operators on vectors that specify the region of integration. If P is an arbitrary path in space broken polygonally into n small vectors $\mathbf{v}_i$, then the integral of the 1-form ω over P is

$$\int_P \omega = \lim_{n\to\infty} \sum_{i=1}^{n} \omega[\mathbf{v}_i] \tag{22}$$

where the vectors $\mathbf{v}_i$ become infinitesimal in the limit. Eq. (22) shows that the graphical representation of forms as operators on vectors extends to integration of forms over paths:

the integral of a 1-form over a path is the number of surfaces of the 1-form pierced by the path.

The interior product is written $\mathbf{v} \lrcorner \omega$ where $\mathbf{v}$ is a vector and ω is a form. If ω is a 1-form, the interior product of $\mathbf{v}$ and ω is

$$\mathbf{v} \lrcorner \omega \equiv \omega[\mathbf{v}] \tag{23}$$

which is simply the definition of ω as a linear operator acting on the vector $\mathbf{v}$.

The interior product of a vector and a 2-form follows from the definition of a 2-form as the antisymmetrized tensor product of two 1-forms, so that $a \wedge b = a \otimes b - b \otimes a$, where a and b are 1-forms and $\otimes$ is the tensor product. The tensor product of 1-forms a and b is defined so that if $\mathbf{v}$ and $\mathbf{w}$ are vectors, $a \otimes b$ is a function of two vectors, and $a \otimes b[\mathbf{v}, \mathbf{w}] = a[\mathbf{v}]b[\mathbf{w}]$, which is a real number. If we operate $a \wedge b$ on one vector $\mathbf{v}$ rather than two, we obtain the 1-form $a[\mathbf{v}]b - b[\mathbf{v}]a$. This is the interior product $v \lrcorner (a \wedge b)$. The interior product of $\hat{x}$ and $dx \wedge dy$ is

$$\begin{aligned} \hat{x} \lrcorner (dx \wedge dy) &= dx \wedge dy[\hat{x}, \cdot\,] \\ &= dx[\hat{x}] \otimes dy - dy[\hat{x}] \otimes dx \\ &= dy. \end{aligned} \tag{24}$$

The interior product of a vector $\mathbf{v}$ and a p-form $a\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ can be obtained for arbitrary p by repeated application of the definition of the exterior product as the antisymmetrized tensor product. When $a\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ is expanded in terms of the tensor product, the first factor of each term operates on the vector $\mathbf{v}$. This can be written conveniently using the determinant,

$$\mathbf{v} \lrcorner \omega = a\, det \begin{vmatrix} dx^{i_1}[\mathbf{v}] & \ldots & dx^{i_p}[\mathbf{v}] \\ dx^{i_1} & \ldots & dx^{i_p} \\ \vdots & & \vdots \\ dx^{i_1} & \ldots & dx^{i_p} \end{vmatrix} \tag{25}$$

where the top row of 1-forms operate on $\mathbf{v}$ and the determinant then evaluates to a $(p-1)$-form.

We can now define the interior product $\alpha \lrcorner \omega$ of a 1-form α and a p-form ω. The metric g_{ij} must be used to convert a 1-form to its dual vector ("raising the index"). With the euclidean metric, a 1-form and its dual have the same components, so that we can write in R^3,

$$(a\,dx + b\,dy + c\,dz) \lrcorner \omega = (a\hat{x} + b\hat{y} + c\hat{z}) \lrcorner \omega. \tag{26}$$

In curved spaces or curvilinear coordinates, the definition becomes

$$\alpha_i\, dx^i \lrcorner \omega = g^{ij} \alpha_j\, dx_i \lrcorner \omega \tag{27}$$

where g^{ij} is the inverse of the metric g_{ij} and repeated indices are summed over $1, \ldots, n$ with n the dimension of space. $g^{ij}\alpha_j\, dx_i$ is the vector dual to the 1-form α. The components of the metric for spherical coordinates are

$$g_{ij} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{bmatrix} \tag{28}$$

which along with (27) allow the boundary conditions given in Sec. 2 to be used for geometries with spherical symmetry.

We note in passing that one can also define the interior product of a p-form and a q-form for arbitrary p and q using the relation [4]

$$v \lrcorner \star w = \star(w \wedge v) \tag{29}$$

where $\star$ is the Hodge star operator. If $\star\star = 1$, as is the case for three-dimensional space with a positive definite metric, this can be used to rewrite the boundary projection operator in terms of the exterior product and star operator as $\star[\star(n \wedge (\alpha_2 - \alpha_1)) \wedge n]$.

In view of (27), the interior product of two 1-forms is simply their inner product. The magnitude $|\alpha|$ of a 1-form α is then $\sqrt{\alpha \lrcorner \alpha}$ or $\sqrt{\alpha \lrcorner \alpha^*}$ where the superscript * denotes complex conjugation if the coefficients of α are complex.

The interior product distributes over the exterior product by.

$$\alpha \lrcorner (\beta \wedge \gamma) = (\alpha \lrcorner \beta) \wedge \gamma + (-1)^p \beta \wedge \alpha \lrcorner \gamma \tag{30}$$

where p is the degree of β.

A.2 The Interior Product in R^3

For R^3 with the euclidean metric the interior products has a simple computational rule. The interior product of a differential dx and a term of an arbitrary form containing dx as a factor is found by moving dx to the left of the term, switching the sign of the term each time two differentials are interchanged, and then removing the differential dx from the term. If dx is not present in the term, the interior product is zero. For example,

$$\begin{aligned} 3\,dx \lrcorner (\,dz \wedge \,dx + 2\,dy \wedge \,dz) &= 3\,dx \lrcorner \,dz \wedge \,dx + 6\,dx \lrcorner \,dy \wedge \,dz \\ &= -3\,dx \lrcorner \,dx \wedge \,dz + 0 \\ &= -3\,dz. \end{aligned} \tag{31}$$

This rule can be used to obtain the interior product of a 1-form and an arbitrary p-form.

As noted above, the interior product of two 1-forms is their inner product,

$$(a_1\,dx + a_2\,dy + a_3\,dz) \lrcorner (b_1\,dx + b_2\,dy + b_3\,dz) = a_1b_1 + a_2b_2 + a_3b_3, \tag{32}$$

The interior product of a 1-form and a 2-form is

$$\begin{aligned} &(a_1\,dx + a_2\,dy + a_3\,dz) \lrcorner (b_1\,dy\,dz + b_2\,dz\,dx + b_3\,dx\,dy) = \\ &\quad (a_3b_2 - a_2b_3)\,dx + (a_1b_3 - a_3b_1)\,dy + (a_2b_1 - a_1b_2)\,dz \end{aligned} \tag{33}$$

which compares to the cross product of vectors with the same components. The exterior product of two 1-forms, the interior product of a 1-form and a 2-form and the vector cross product result in the same coefficients. The exterior product of a 1-form and a 2-form, the interior product of two 1-forms and the vector inner product result in the same coefficient. With this identification of operations on forms with vector operations, the rule (30) for R^3 contains several vector identites as special cases.

A.3 Examples of the Use of the Interior Product

The vector Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{34}$$

where $\mathbf{v}$ is the velocity of the charge q. Written using differential forms, this law becomes

$$F = q(E - \mathbf{v} \lrcorner B) \tag{35}$$

where F is the force field 1-form. The vector current density is given by

$$\mathbf{J} = \mathbf{v}q \tag{36}$$

where $\mathbf{v}$ is the velocity field of a volume charge distribution with scalar density q. Writing the same charge distribution as the 3-form ρ, this becomes

$$J = \mathbf{v} \lrcorner \rho. \tag{37}$$

If $\rho = q\,dx\,dy\,dz$ and the flow field $\mathbf{v} = v\hat{x}$, for example, then $J = v\hat{x} \lrcorner q\,dx\,dy\,dz = vq\,dy\,dz$.

A.4 Twisted Forms

This appendix is intended only to provide background in the mathematics of twisted forms for the interested reader. In practice, we agree to use only right-handed coordinates, and the distinction between twisted and nontwisted forms disappears. One can switch between outer and inner orientations at will, using whichever is more appropriate to the quantity at hand. The use of twisted forms in EM theory therefore does not complicate calculations; rather, twisted forms become a graphical tool, adding significantly to the intuition obtainable from pictures of boundary conditions.

A twisted form changes sign under a reflection of the coordinate system relative to a nontwisted form with the same components. Other terms for twisted tensors are *oriented*, *Weyl* or *odd*. Axial vectors or pseudovectors are dual to twisted 1-forms and nontwisted 2-forms in R^3; polar vectors are dual to nontwisted 1-forms and twisted 2-forms.

Graphically, each differential form is represented by surfaces in space. There are two possible orientations for each set of surfaces, and so an orientation must be specified in addition to the surfaces. A nontwisted form has an inner orientation, so that each of its surfaces has a perpendicular direction associated with it. The nontwisted 1-form dx has inner orientation in the $+x$ direction; the 2-form $dy\,dz$ obtains a screw sense from the inner orientations of dy and dz.

A twisted p-form is given an outer orientation rather than an inner orientation. Graphically, an outer orientation for a form α is the orientation of a form consisting of surfaces orthogonal to the surfaces of α. In Burke's [6] notation, a twisted form is written as a pair (α, Ω) of a nontwisted form α and volume element (n-form) Ω. If the twisted form α has degree p, its outer orientation $\{(\alpha, \mathit{\Omega})\}$ is an $(n-p)$-form and is given by

$$\{(\alpha, \mathit{\Omega})\} \wedge \alpha = \Omega \tag{38}$$

Under coordinate reflection, the volume element changes sign, so that outer orientations also must change sign relative to inner orientations. Note that the sign change is relative; under reflection, the orientation of a nontwisted form may reverse while the corresponding twisted form retains its original orientation. This is the case for 2-forms in R^3.

Fig. 4a shows the nontwisted 1-form $dx + 2\,dy$ with inner orientation given by an arrow. Fig. 4b shows the twisted form $dx + 2\,dy$ in R^2. Its outer orientation with respect to $\Omega = dx\,dy$ is provided by the 1-form $\frac{1}{5}(2\,dx - dy)$, which has orientation in the direction of the arrow in Fig. 4b. Under coordinate reflection, the orientation of a twisted form reverses with respect to the orientation of the nontwisted form with the same components, as shown in the figure.

Under pullback or restriction to a subspace, the outer orientation[1] $\{(\alpha_s, \Omega_s)\}$ of the twisted form $\{(\alpha, \Omega)\}$ restricted to a subspace with outer orientation n,

$$\{(\alpha_s, \Omega_s)\} \wedge n = \{(\alpha, \Omega)\} \tag{39}$$

where α_s is the restriction of the form α to a subspace and Ω_s is a volume element in the subspace. The 1-form $n = \frac{df}{\sqrt{df \lrcorner df}}$ used in the preceeding sections serves as the outer orientation for the boundary $f = 0$, so that Eq. (39) and the definition (38) lead to the convention (10).

Twisted forms are the natural mathematical object to represent sources. Consider, for example, a surface current in the $+x$ direction along the $x - y$ plane. The usual vector $\mathbf{J}_s$ for this current is $J_0\hat{x}$. The nontwisted 1-form $J_0\,dx$ dual to $J_0\hat{x}$ has inner orientation in the $+x$ direction, but does not integrate over a path to yield the correct current through the path—the current through the y-axis from $y = 0$ to $y = 1$, for example, is J_0, but the integral of $J_0\,dx$ over that path is zero. The nontwisted 1-form $J_0\,dy$ integrates properly to yield current through a path, but has inner orientation in the $+y$ direction. The twisted 1-form $(J_0\,dy,\ dx\,dy)$ by (38) has outer orientation dx, which is the direction of current flow, and integrates to yield the correct current through an arbitrary path as well.

Twisted 2-forms are similarly useful for representing surface charge. Each cell of a nontwisted 2-form in two-space has a screw sense as inner orientation, whereas each cell of a twisted 2-form has a sign as its outer orientation (Fig. 5).

We represent surface current and surface charge density with twisted forms because of the graphical convenience of having orientation of forms correspond to actual direction of flow of current or sign of the charge. To be precise, H and D must be twisted forms also. For clarity's sake, we ignore this in the body of the paper, since as mentioned above one can employ inner and outer orientations interchangeably if only right-handed coordinates are used. It is interesting to note that when properly formulated using both twisted and nontwisted forms as in [5], the $3+1$ representation of electromagnetic field theory becomes explicitly parity invariant—no right-hand rule is required.

References

[1] G. A. Deschamps, "Electromagnetics and differential forms," *IEE Proceedings*, vol. 69, no. 6, pp. 676–696, June 1981.

[2] D. Baldomir, "Differential forms and electromagnetism in 3-dimensional Euclidean space R^3," *IEE Proceedings*, vol. 133, no. 3, pp. 139–143, May 1986.

[3] N. Schleifer, "Differential forms as a basis for vector analysis—with applications to electrodynamics," *American Journal of Physics*, vol. 51, no. 12, pp. 1139–1145, Dec. 1983.

[4] W. Thirring, *Classical Field Theory*. New York: Springer-Verlag, 1978.

[5] W. L. Burke, "Manifestly parity invariant electromagnetic theory and twisted tensors," *Journal of Mathematical Physics*, vol. 24, no. 1, pp. 65–69, Jan. 1983.

[6] W. L. Burke, *Applied Differential Geometry*. Cambridge: Cambridge University Press, 1985, pp. 79-80, 183-200, 192-193, 279-282.

[7] P. Bamberg and S. Sternberg, *A Course in Mathematics for Students of Physics*, vol. II. Cambridge: Cambridge University Press, 1 ed., 1988, pp. 621-626.

[8] C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation*. San Francisco: Freeman, 1973, pp. 90-129.

Affiliation of Authors. The authors are with the Department of Electrical and Computer Engineering, 459 Clyde Building, Brigham Young University, Provo, UT, 84602.

Footnotes

1. This definition for the orientation of the restriction of a form to a boundary, or pullback to the boundary, corresponds with the expected orientations of EM boundary conditions. For pullback to commute with exterior differentiation, however, the convention $n \wedge \{(\alpha_s, \Omega_s)\} = \{(\alpha, \Omega)\}$ found in Burke, [6] must be used instead of Eq. (39) (Burke, W. L., Private communication, Feb. 1995). This convention could be employed here if the surface normal 1-form were oriented from D_1 to D_2 as before but from H_2 to H_1. In fact, applying the continuity equation for surface charge and the commutation of pullback with exterior differentiation to Burke's boundary conditions using pullback, one can show that the pullback functions for H must differ from those for D by a sign. We prefer to leave the boundary conditions simple at the expense of some required caution when using our definition of pullback more generally.

Fig. 1. (a) A boundary with discontinuous magnetic field. (b) The field $(H_2 - H_1)$. (c) The 1-form $n \lrcorner (n \wedge (H_2 - H_1))$. (d) The restriction of this 1-form to the boundary, along with the corresponding vector $\mathbf{J}_\mathrm{s}$.

Fig. 2. (a) A boundary with discontinuous electric flux. (b) The field $(D_2 - D_1)$. (c) The 2-form $n \lrcorner (n \wedge (D_2 - D_1))$. (d) The restriction of this 2-form to the boundary.

Fig. 3. (a) The boundary $-\cos y + z = 0$ with magnetic field $H_2 = H\,dx$ above the boundary. The field is zero below the boundary. (b) The 1-form $H\,dx$ in 3-space. (c) The 1-form $H\,dx$ restricted to the boundary.

Fig. 4. (a) The behavior of the nontwisted 1-form $dx + 2\,dy$ in the plane under a coordinate transform. (b) The behavior of the twisted 1-form $dx + 2\,dy$ under the same transform. The orientations of the untransformed forms are related by the right-hand rule. Since the coordinate transform changes the handedness of the coordinate system, the orientations of the transformed forms are related by the left-hand rule, and so the orientation of the twisted form reverses.

Fig. 5. (a) The inner orientation of a nontwisted 2-form β in two-space, specified by a screw sense for each box. (b) The outer orientation of a twisted 2-form (β, Ω) in two-space is a sign for each box.

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.