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Electromagnetic Green functions using differential forms

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Abstract–In this paper we redevelop the scalar and dyadic Green functions of electromagnetic theory using differential forms. The Green dyadic becomes a double form, which is a differential form in one space with coefficients that are forms in another space, or a differential form-valued form. The results presented here correspond closely with the usual dyadic treatment, but are clearer and more intuitive. Many of the usual expressions using green functions in vector notation require a surface normal; with the Green forms the surface normal is unnecessary. We illustrate the formalism by computing scattering from a randomly rough conducting surface and deriving the Green form for a dielectric half-space. We also define the interior derivative, which is equivalent to the coderivative but for a constant metric has a computational rule dual to that of the exterior derivative and simplifies calculation in coordinates. This work makes available some of the tools that have not yet been presented in the language of differential forms but are essential in applied electromagnetics.

1. INTRODUCTION

In this paper we treat Green function methods in electromagnetic (EM) field theory using the calculus of differential forms. The calculus of differential forms has been applied to EM theory by Deschamps [1], Baldomir [2], Schleifer [3], Thirring [4], Burke [5,6], Bamberg [7], Ingarden and Jamiolkowski [8], Parrott [9] and others.

Several authors have advocated the use of the calculus of forms in engineering EM theory, but some important tools for applied problems have not been developed. In [10] the authors presented a representation of EM boundary conditions using differential forms. In this work we develop another tool well suited for practical use, the Green form in the $(3+1)$ representation.

As proposed by Thirring [4], the EM Green function becomes a double form. Double forms are defined by de Rham in [11]. Green forms are treated in the mathematics literature (see [12] and its references), and Thirring gives the time-dependent Green form for electrodynamics in Minkowski spacetime. Our Green form has the same components as the Green dyadic in Kong [13] and therefore is easily related to the usual methods in applied electromagnetics. We derive expressions for the Green double form in terms of the scalar Green function, the electric field due to a surface current density and the Stratton-Chu formula.

The use of differential forms makes the results presented here clearer in certain ways than the usual vector and dyadic treatment. In obtaining expressions for observed fields in terms of the Green forms, the product rule for the exterior derivative takes the place of several vector identities. This makes the derivation much cleaner. The dyadic expression for observed fields due to tangential fields along the surface of an observation region using the Green dyadic includes a surface normal. With the corresponding expression using the Green form, the surface normal is unnecessary.

In this paper we also define the interior derivative, which is equivalent to the standard coderivative [7], but simplifies calculations in coordinates when the metric is constant. The computational rule which we propose is dual to that of the exterior derivative.

In Section 2 we review operations on forms and treat double forms briefly. In Section 3 we solve Maxwell's laws of electromagnetics in terms of the Green double form and the scalar Green function. Finally, in Section 4 we illustrate the method by computing scattered fields from a rough conducting surface and deriving the Green form for a dielectric half-space. This work shows that the calculus of differential forms can be used in all applications to which Green functions and dyadics are suited.

2. DEFINITIONS

In this section we give definitions and notation to be used in Section 3 to derive the Green forms. We define the interior and exterior derivatives, the interior and exterior products and the Laplace-de Rham operator. Double 1-forms are also introduced in this section.

2.1 Operators

The exterior derivative d is defined in [7] and elsewhere. It can be represented formally by

$$d \equiv \frac{\partial}{\partial x^i}\, dx^i \wedge \tag{1}$$

where $x^1 \ldots x^n$ are coordinates on an n-dimensional space and the summation convention is used. The exterior product $\wedge$ is the antisymmetrized tensor product, so that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and $dx^i \wedge dx^i = 0$ (Often the wedge between differentials is dropped; there is an implied wedge between the differentials in the integrand of any multiple integral.) The partial derivatives $\frac{\partial}{\partial x^i}$ act on the coefficients of a form, so that in R^3, $d(f\, dx) = \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx$ since $dx \wedge dx = 0$. We define the interior derivative in the euclidean metric similarly,

$$d \lrcorner \equiv \frac{\partial}{\partial x^i} dx^i \lrcorner \tag{2}$$

where $\lrcorner$ is the interior product defined in [10]. In R^3 we have $d\lrcorner\,(f\, dx) = \frac{\partial f}{\partial x}$ since $dx \lrcorner dx = 1$ and $dy \lrcorner dx = dz \lrcorner dx = 0$.

The interior product is defined to be the contraction of a vector with a k-form (which is a totally antisymmetric $\binom{0}{k}$ tensor). In this paper we use the euclidean metric, so we can extend this definition to the interior product of a 1-form and k-form easily since vectors and 1-forms have the same components. The interior product of dx^{i_j} and $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is zero for i_j not equal to any of $i_1 \ldots i_k$, otherwise it is $(-1)^{j-1} dx^{i_1} \wedge \cdots \wedge dx^{i_{j-1}} \wedge dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_k}$ for $1 \le j \le k$. Thus, by (2) the interior derivative of a form is computed by moving each differential in turn to the leftmost position by alternating the sign of the form each time two differentials are swapped, removing that differential and taking the corresponding partial derivative.

The interior derivative is equivalent[1] up to a sign to the coderivative defined in [4, 7] and elsewhere,

$$d\,\lrcorner\,\omega = (-1)^{k+1} \star^{-1} d \star \omega \tag{3}$$

where k is the degree of ω and $\star$ is the Hodge star operator. In R^3 with the euclidean metric, $\star 1 = dx\,dy\,dz$, $\star dx = dy \wedge dz$, $\star dy = dz \wedge dx$, $\star dz = dx \wedge dy$ and $\star^{-1} = \star$. Note that the interior derivative contains the sign $(-1)^{k+1}$ naturally. For a nonconstant metric, such as would arise in curvilinear coordinates, (3) replaces (2) as the definition of the interior derivative.

The interior derivative is easier to compute with than the co-derivative, as illustrated by the following example. We first use the coderivative to find

$$\begin{aligned}
&- \star d \star (D_1\, dy\, dz + D_2\, dz\, dx + D_3\, dx\, dy) \\
&= - \star \Big(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz\Big) \wedge \Big(D_1\, dx + D_2\, dy + D_3\, dz\Big) \\
&= - \star \Big(D_{1x}\, dx \wedge dx + D_{1y}\, dy \wedge dx + D_{1z}\, dz \wedge dx \\
&\qquad + D_{2x}\, dx \wedge dy + D_{2y}\, dy \wedge dy + D_{2z}\, dz \wedge dy \\
&\qquad + D_{3x}\, dx \wedge dz + D_{3y}\, dy \wedge dz + D_{3z}\, dz \wedge dz\Big) \\
&= \Big(D_{2z} - D_{3y}\Big) dx + \Big(D_{3x} - D_{1z}\Big) dy + \Big(D_{1y} - D_{2x}\Big) dz
\end{aligned}$$

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[1] We prefer the term "interior derivative" to the usual "coderivative" for pedagogical reasons.

The definition (2) also provides a computational advantage over (3) when extended to a noneuclidean (but constant) metric, such as would arise when modeling material properties. The interior product is defined to be the contraction of a vector and p-form. Formally, the index of the "1-form" $\frac{\partial}{\partial x^i} dx^i$ must be raised using the inverse metric g^{ij}, so that the interior derivative becomes for a constant but otherwise arbitrary metric,

$$d\,\lrcorner = g^{ij} \frac{\partial}{\partial x^i}\, dx_j\,\lrcorner$$

where the dx_j are a basis for the vector space dual to the space of 1-forms. It can be shown that this expression is equivalent to the coderivative for the same metric (the star operator is metric-dependent; see [4]). A non-reciprocal medium yields a non-symmetric g^{ij}, which is not a metric, but can still be used in the definition of the interior derivative.

Federer [14] also defines an interior derivative, but it takes p-vectors to $(p-1)$-vectors and is metric independent.

Using the definition of the interior derivative we compute the same result immediately,

$$d\lrcorner(D_1\,dy\,dz + D_2\,dz\,dx + D_3\,dx\,dy) = D_{1y}\,dz - D_{1z}\,dy + D_{2z}\,dx - D_{2x}\,dz + D_{3x}\,dy - D_{3y}\,dx$$

where we have removed each differential in turn, after it moving to the left if necessary, and taken the corresponding partial derivative.

The Laplace-de Rham operator Δ is

$$\Delta = d\lrcorner d + dd\lrcorner \tag{4}$$

which is a generalization of the vector operator ∇^2. With the euclidean metric, Δ becomes $(\Delta\omega)_i = \sum_j \frac{\partial^2}{\partial x_j^2}\omega_i$ where the subscript i indexes components of ω. On 1-forms, (4) is equivalent to the euclidean vector identity $\nabla^2 = -\nabla\times\nabla\times + \nabla\nabla$.

The generalized Stokes theorem is

$$\int_V d\omega = \int_{\partial V} \omega \tag{5}$$

where ω is a p-form and V is a $p+1$ dimensional region with ∂V as its boundary. Also, the interior product of two arbitrary forms a and b satisfies

$$a\lrcorner b = \star(\star b \wedge a) \tag{6}$$

where $\star$ is the Hodge star operator.

2.2 Double Forms

A double form [11] is a differential form in one space with coefficients that are forms in another space. The double forms that we will use in this paper are associated with $R^3 \times R^{3'}$ where R^3 is the observation space and $R^{3'}$ is the source space. We will use 1-form valued 1-forms, or double 1-forms, which can be written in general

$$\begin{aligned} G = {} & G_{11}\,dx\,dx' + G_{12}\,dx\,dy' + G_{13}\,dx\,dz' \\ & + G_{21}\,dy\,dx' + G_{22}\,dy\,dy' + G_{23}\,dy\,dz' \\ & + G_{31}\,dz\,dx' + G_{32}\,dz\,dy' + G_{33}\,dz\,dz' \end{aligned}$$

Between the primed and unprimed differentials there is an implied tensor product rather than an exterior product. The coefficients are functions $G_{ij}(\mathbf{r},\mathbf{r}')$ of both the observation and source coordinates.

A double form can be used as a transformation kernel (if its coefficients vanish sufficiently quickly at infinity). If we fix a double 1-form G, we have the transformation from $R^{3'}$ to R^3 given by the volume integral

$$\omega = \int G \wedge \star\omega' \tag{7}$$

where ω' is a 1-form and ω is the transform of ω' due to the kernel G. The exterior product yields a 3-form in $dx'\,dy'\,dz'$ which is integrated over $R^{3'}$. The unprimed differentials remain, resulting in the 1-form ω in the observation space.

Components of a dyadic are of the form $\hat{x}^i\,\hat{x}^j$ with no prime, which does not show explicitly the relationship the dyadic can provide to the source and observation spaces. The action of the double form G, as a kernel from the source to the observation space is clearly reflected in the product of primed and unprimed differentials in each component.

We introduce the identity kernel δI where I is the double form

$$dx\,dx' + dy\,dy' + dz\,dz' \tag{8}$$

or $dr\,dr' + rd\theta\,r'd\theta' + r\sin\theta\,d\phi\;r'\sin\theta' d\phi'$ in spherical coordinates. δ is the three-dimensional Dirac delta function $\delta(x-x')\,\delta(y-y')\,\delta(z-z')$. Using this kernel, $\int \delta I \wedge \star\omega' = \omega'|_{(x,y,z)} = \omega$, so that the transformation takes ω' from source to observation space without otherwise changing its components.

3. THE EM GREEN FORMS

In this section we derive expressions for the electric field 1-form at an observation point due to applied sources and fields using the Green forms. We consider time-harmonic ($e^{-i\omega t}$) fields in an isotropic medium of permittivity ϵ and permeability μ.

We write Faraday's and Ampere's laws as

$$dE = i\omega B \tag{9}$$

$$d\lrcorner B = i\omega\epsilon\mu E - \mu \star J \tag{10}$$

where E is the electric field intensity 1-form, B is the magnetic flux density 2-form and J is the electric current density 2-form. The constitutive relations are $D = \epsilon \star E$ and $B = \mu \star H$, where D is the electric flux density 2-form and H is the magnetic field intensity 1-form.

By taking the interior derivative of (9) and substituting (10), we obtain

$$(d\lrcorner d + k^2)E = -i\omega\mu \star J \tag{11}$$

where $k^2 = \omega^2\mu\epsilon$. The Green double 1-form G for Eq. (11) then satisfies

$$(d\lrcorner d + k^2)G = -\delta I. \tag{12}$$

Here and below, all derivatives will act on primed coordinates unless otherwise noted, but to avoid clutter, the derivatives will remain unprimed.

Let V' be a volume containing source current density given by the 2-form J'. Outside V', the electric field due to the sources is E, a 1-form in the observation space. Using the unit kernel δI we can write

$$E = \int_{V'} \delta I \wedge \star E' \tag{13}$$

Substituting Eq. (12) into (13), we obtain

$$E = \int_{V'} [d \star dG \wedge E' - k^2 G \wedge \star E'] \tag{14}$$

where we have used $d\lrcorner dG = (-1)^1 \star d \star dG$ and moved a $\star$ across the exterior product (if λ and ν are both p-forms, then $\star\lambda \wedge \nu = \lambda \wedge \star\nu$, as can be verified easily in coordinates). Using the product rule for the exterior derivative, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg\,\alpha}\alpha \wedge d\beta$, Eq. (14) becomes

$$E = \int_{V'} [G \wedge (d \star dE' - k^2 \star E') + d(\star dG \wedge E' + G \wedge \star dE')]. \tag{15}$$

After applying the star operator to Eq. (11) and using the definition of the interior derivative, we can insert J' into (15),

$$E = \int_{V'} [i\omega\mu G \wedge J' + d(\star dG \wedge E' + G \wedge \star dE')]. \tag{16}$$

Applying the generalized Stokes theorem and using Faraday's law, we find that

$$E = i\omega\mu \int_{V'} G \wedge J' + \int_{\partial V'} (i\omega\mu G \wedge H' + \star dG \wedge E') \tag{17}$$

where the second term takes into account fields on the surface $\partial V'$ due to sources outside of V'.

The integrals in (17), like all integrals of differential forms, can be integrated by the method of pullback [7]. This method is completely general and allows forms to be integrated conveniently over parameterized regions.

The expression corresponding to (17) using the dyadic Green function $\overline{\overline{G}}$ is

$$\mathbf{E} = i\omega\mu \int_{V'} \overline{\overline{G}} \cdot \mathbf{J}' dv' - \int_{\partial V'} \left(i\omega\mu \overline{\overline{G}} \cdot [\hat{n} \times \mathbf{H}'] + \nabla \times \overline{\overline{G}} \cdot [\hat{n} \times \mathbf{E}'] \right) ds'$$

where $\hat{n}$ is an outward surface normal. Because E' and H' are 1-forms rather than 2-forms, their exterior product with G behaves differently than the exterior product $G \wedge J'$. Components of E' and H' tangent to $\partial V'$ naturally do not contribute to the surface integral in (17). Thus, the surface normal is eliminated and a simpler expression results. Also, Green's theorem is used in deriving the dyadic result. Green's theorem on forms is an immediate consequence of the product rule for the exterior derivative and the generalized Stokes theorem.

3.1 The Scalar Green Function

The scalar Green function g satisfies the wave equation for an elementary source $-\delta$,

$$(\Delta + k^2) g = -\delta \tag{18}$$

It can easily be shown that gI satisfies Eq. (18)) for the source $-\delta I$. Substituting gI for g and $-\delta I$ for $-\delta$ in (18), expanding the Laplace-de Rham operator and rearranging gives

$$d \lrcorner d(gI) + k^2 (gI + \frac{1}{k^2} dd \lrcorner gI) = -\delta I \tag{19}$$

Since $dd = 0$, this can be rewritten as

$$d \lrcorner d(gI + \frac{1}{k^2} dd \lrcorner gI) + k^2 (gI + \frac{1}{k^2} dd \lrcorner gI) = -\delta I \tag{20}$$

By comparison with Eq. (21), we see that

$$G = (1 + \frac{1}{k^2} dd \lrcorner) gI \tag{21}$$

up to a solution of $(d \lrcorner d + k^2) H = 0$ where H is a double 1-form. This freedom is used to satisfy boundary conditions. The double 1-form I could be included in the definition of g, but we prefer to leave it out so that g is the usual scalar Green function.

By pullback we can transform Eq. (18) to the spherical coordinate system. Noting that g_0 in free space is spherically symmetric, we find that

$$\frac{1}{r}\frac{d^2}{dr^2}rg_0 + k^2 g_0 = -\delta(r) \tag{22}$$

The solution of this differential equation is the usual result

$$g_0 = \frac{e^{ikr}}{4\pi r} \tag{23}$$

From this we can compute G_0,

$$G_0 = g_0 I + \frac{1}{k^2} dd\lrcorner(g_0 I) \tag{24}$$

which becomes $g_0(rd\theta\, r'd\theta' + r\sin\theta d\phi\, r'\sin\theta' d\phi')$ in the far field.

It is easily verified that if g is symmetric in $\mathbf{r}$ and $\mathbf{r}'$ (as is the case for reciprocal media), the derivatives in the first term of Eq. (21) can be taken to act on umprimed rather than primed coordinates. Thus, we can write using (17) after neglecting sources outside of V',

$$E = i\omega\mu(1 + \frac{1}{k^2}dd\lrcorner)\int_{V'} gI \wedge J \tag{25}$$

where the derivatives act on unprimed coordinates. The Lorentz gauge is $d\lrcorner A = i\omega\epsilon\mu\phi$, where ϕ is the scalar electric potential and A is the magnetic vector potential 1-form. In the Lorentz gauge, $E = i\omega A - d\phi$ together with (25) imply that $A = \mu\int_{V'} gI \wedge J$.

For a region V' containing no sources, Eq. (17) becomes

$$E = \int_{\partial V'} (i\omega\mu G \wedge H' + \star dG \wedge E' \tag{26}$$

Substituting (21) and using $dd = 0$ gives

$$E = \int_{\partial V'} (i\omega\mu gI \wedge H' + \frac{i\omega\mu}{k^2} dd\lrcorner gI \wedge H' + \star dgI \wedge E') \tag{27}$$

Using the product rule for the exterior derivative, (27) can be rewritten as

$$E = \int_{\partial V'} \left(i\omega\mu gI \wedge H' + \frac{i\omega\mu}{k^2}\left[d(d\lrcorner gI \wedge H') - d\lrcorner gI \wedge dH'\right] + \star dgI \wedge E' \right) \tag{28}$$

The second term vanishes by the generalized Stokes theorem. Using Ampere's law, we obtain the Stratton-Chu formula,

$$E = \int_{\partial V'} \left[i\omega\mu gI \wedge H' + (\star dgI) \wedge E' - (d\lrcorner gI) \wedge \star E' \right] \tag{29}$$

which again eliminates the dot and cross products with a surface normal found in the usual vector expression [13].

3.2 Fields Due to a Surface Current

We can find the observation fields due to a surface current using either the first or the second term of Eq. (17). We will do the computation both ways. The first method illustrates how the direction normal to a surface can be integrated out using the interior product and a singular integrand. The second method uses the boundary conditions in [10] to arrive at the same result.

Suppose that the electric current density is given by $J'\delta(f)$ where the differentiable function $f(x', y', z')$ is zero only along the surface $\partial V'$, J' is a 2-form parallel to $\partial V'$ and $\delta(f)$ is the Dirac delta function $\sqrt{df \lrcorner df}\, \delta(f(x', y', z'))$. Graphically, the surfaces of the 1-form df and the surface normal 1-form $n = \frac{df}{\sqrt{df \lrcorner df}}$ are parallel to $\partial V'$ because f is constant along $\partial V'$. The tubes of the 2-form J' are also parallel to $\partial V'$, so that J' must contain a factor n. We can thus decompose J' into the product $n \wedge T$ where T is a 1-form satisfying $n \lrcorner T = 0$. The interior product distributes over the exterior product by $\alpha \lrcorner (\beta \wedge \gamma) = (\alpha \lrcorner \beta) \wedge \gamma + (-1)^{deg \beta} \beta \wedge (\alpha \lrcorner \gamma)$. It follows that $n \wedge (n \lrcorner J') = J'$. Substituting this into the first term of (17), we find that

$$E = i\omega\mu \int_{V'} G \wedge (n \wedge n \lrcorner J')\delta(f)$$

Integrating along the direction perpendicular to $\partial V'$ eliminates $n\delta(f)$, so that

$$E = i\omega\mu \int_{\partial V'} G \wedge J_s' \tag{30}$$

where J_s is the surface current 1-form defined by $\delta(f) J_s = n \lrcorner J$ (see reference [10]; the surface current can be defined equivalently to be $J_s = -\mathbf{v} \lrcorner \rho_s$ where $\mathbf{v}$ is the velocity field of the surface charge density 2-form ρ_s).

Alternately, if there is a tangential magnetic field H' along the boundary of V', we can can place by equivalence a surface current density $J'\delta(f)$ on $\partial V'$ satisfying the appropriate boundary condition for vanishing magnetic field inside V'. Since the integral in the second term of (17) is along $\partial V'$, we can apply the boundary projection operator $n \lrcorner n \wedge$ to the integrand without affecting the integral,

$$E = i\omega\mu \int_{\partial V'} n \lrcorner (n \wedge G \wedge H').$$

After interchanging n and G using the antisymmetry of the exterior product, we distribute the interior product $n \lrcorner$ to obtain

$$E = -i\omega\mu \int_{\partial V'} \{(n \lrcorner G) \wedge n \wedge H' - G \wedge [n \lrcorner (n \wedge H')]\} \tag{31}$$

Since the 2-form $n \wedge H'$ is parallel to the surface $\partial V'$ and $n \lrcorner G$ is a 0-form in the primed coordinates, the first term of (31) drops out. By the boundary condition $n \lrcorner (n \wedge H') = J_s'$ the second term reduces to Eq. (30).

There is an interesting difference between Eq. (30) and the usual vector result. In the integral $\int_{\partial V'} \overline{\overline{G}} \cdot \mathbf{J_s}'\, ds'$, the vector $\mathbf{J_s}'$ must lie on the surface $\partial V'$ or the integration yields an incorrect result. In (30), J_s can be arbitrary because its normal component does not contribute to the integral.

As noted in [10], the surface current density 1-form differs from the usual vector quantity $\mathbf{J_S}$ in that it includes the geometry of the boundary more naturally, making the computation of fields due to surface sources using the Green form slightly simpler than with vectors.

4. APPLICATIONS

In this section we give two elementary applications of the Green form as exercises in the use of this method: on-axis scattering from a rough surface and the derivation of the Green form for a dielectric half-space. The first example illustrates the simplification that results due to the absence of surface normals in the Green form expressions. The second is intended as a reference to show how Green forms are manipulated in solving a standard problem.

4.1 Rough Surface Scattering

Consider a rough, perfectly conducting surface $z = \xi(x, y)$. For a horizontally polarized plane wave propagating in the $-z$ direction towards the surface with the magnetic field $H_i = \frac{E_0}{\eta_0} e^{-ikz} dy$, the approximate (physical optics) total tangential field at the boundary is

$$H_t' = 2H_{it}|_{z=\xi(x',y')} \tag{32}$$

Since H' in Eq. (17) need not be tangential to $\partial V'$, we can substitute $2H_i'$ in place of $2H_{it}'$. Using the free-space scalar Green function $g_0 = \frac{e^{ik(z-z')}}{4\pi z}$ for the scattered electric field in the z direction, the Green double form G becomes $(I - dz\, dz')g_0$. The third term of (17) drops out since E_t' is approximately zero, so that

$$\begin{aligned} E(z) =& i\omega\mu \int G \wedge H' \\ =& i\omega\mu \int (dx\, dx' + dy\, dy') \frac{e^{ik[z-\xi(x',y')]}}{4\pi z} \wedge \frac{2E_0}{\eta_0} e^{-ik\xi(x',y')} dy' \\ =& \frac{ikE_0 e^{ikz}}{2\pi z} dx \int dx'\, dy' e^{-i2k\xi(x',y')} \end{aligned}$$

where the integration is over the region of illumination.

Obtaining this result using vector notation requires the computation of a surface normal. The surface normal contains a factor due to the curvature of the surface that cancels a factor in the differential surface area element dS. The present method eliminates the need to take into account such geometrical factors that are ultimately extraneous to the problem. While this is only a slight simplification computationally, it shows that the calculus of forms is not only equivalent to vector analysis but is more natural for these types of problems.

4.2 Half-Space Green Forms

Following the dyadic treatment in Kong [13], we determine the Green form G_{10} from source point in region 0, and observation point in region 1. Taking the three-dimensional Fourier transform of $g = \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{4\pi|\mathbf{r}-\mathbf{r}'|}$ and integrating over k_z, we find that for $z < 0$,

$$g(\mathbf{r}) = \frac{i}{(2\pi)^2}\int dk_x\,dk_y \frac{1}{2k_{0z}} e^{i(k_x x + k_y y - k_{0z} z)} \tag{33}$$

where $k_{0z} = \sqrt{k^2 - k_x^2 - k_y^2}$ and the source is at $\mathbf{r}' = 0$. Now, if $\mathbf{K} = \hat{x}k_x + \hat{y}k_y - \hat{z}k_{0z}$ and $K = k_x\,dx + k_y\,dy - k_{0z}\,dz$, then

$$\begin{aligned} dd\lrcorner e^{i\mathbf{K}\cdot\mathbf{r}}(dxdx' + dydy'dzdz') =& d\left[i(k_x\,dx' + k_y\,dy' - k_{0z}\,dz')e^{i\mathbf{K}\cdot\mathbf{r}}\right] \\ =& d\left[iK' e^{i\mathbf{K}\cdot\mathbf{r}}\right] \\ =& -KK' e^{i\mathbf{K}\cdot\mathbf{r}} \end{aligned}$$

Using this result to apply $G = (1 + \frac{1}{k^2} dd\lrcorner) gI$ to Eq. (33), we find that

$$G(\mathbf{r}) = \frac{i}{8\pi^2}\int dk_x\,dk_y \left(I - \frac{KK'}{k^2}\right)\frac{e^{i\mathbf{K}\cdot\mathbf{r}}}{k_{0z}} \tag{34}$$

In order to simplify this expression, we note that I can be rewritten in the new orthonormal basis of 1-forms

$$\begin{aligned} w_e =& \star\frac{K\wedge dz}{|K\wedge dz|} \\ =& \frac{k_y\,dx - k_x\,dy}{\sqrt{k_x^2 + k_y^2}} \\ w_h =& \star\frac{w_e \wedge K}{|K|} \\ =& \frac{k_{0z}(k_x\,dx + k_y\,dy)}{k\sqrt{k_x^2 + k_y^2}} + dz\frac{\sqrt{k_x^2 + k_y^2}}{k} \\ w_k =& \frac{K}{|K|} \end{aligned}$$

The source basis w'_e, w'_h, w'_k is defined similarly but with primed differentials. In this basis, $I - \frac{KK'}{k^2}$ becomes $w_e\, w'_e + w_h\, w'_h$. Translating the source to $\mathbf{r}'$ we have the Green double form from the source region to region 1,

$$\begin{aligned} G_{10}(\mathbf{r},\mathbf{r}') = \frac{i}{8\pi^2}\int dk_x\,dk_y & \frac{e^{i(\mathbf{K}_1\cdot\mathbf{r} - \mathbf{K}\cdot\mathbf{r}')}}{k_{0z}} \\ & [T^{TE}\, w_e(k_{1z})\, w'_e(k_{0z}) + T^{TM}\, w_h(k_{1z})\, w'_h(k_{0z})] \end{aligned} \tag{35}$$

where $\mathbf{K}_1 = \hat{x}k_x + \hat{y}d_y - \hat{z}k_{1z}$. The coefficients T^{TE} and T^{TM} are found by matching the tangential components $dz\lrcorner dz \wedge G$ and $\frac{1}{\mu}dz\lrcorner dz\wedge\star dG$, where $\star d$ acts on unprimed coordinates, of G_{00} and G_{10} at the boundary of regions 0 and 1. Using stationary phase to evaluate the Fourier integral as $kr \to \infty$,

$$\begin{aligned} & G_{10}(\mathbf{r},\mathbf{r}') = \\ & \frac{e^{ikr}}{4\pi r} e^{-\mathbf{K}\cdot\mathbf{r}'}[T_{10}^{TE}\, w_e(k_{1z})\, w'_e(k_{0z}) + \frac{k_1}{k}\, T_{10}^{TM}\, w_h(k_{1z})\, w'_h(k_{0z})] \end{aligned} \tag{36}$$

which is the desired result. Writing the current density 2-form J' in the source basis shows that terms in $w'_h \wedge w'_k$ produce TE waves, whereas terms in $w'_k \wedge w'_e$ produce TM waves.

5. CONCLUSION

In this paper we show that the dyadic Green function can be replaced by a double form. The derivations presented here are more straightforward than their vector-dyadic counterparts because the product rule for the exterior derivative and the generalized Stokes theorem replace unwieldy vector identities. The expressions obtained are also simpler than those using vectors and dyadics because surface normals required in the vector formulation are absent.

Over the past few decades, various authors have contended that the calculus of differential forms brings greater clarity and conciseness to basic electromagnetic theory than vector analysis. With vector analysis, one must force the geometry of the physics into a much smaller set of quantities (scalars, vectors and dyadics) than is available with differential forms. With the geometry of a problem more naturally represented by differential forms, the expressions themselves often suggest the next step in a derivation or lead to a useful physical interpretation. We find that this clarity and conciseness extends to the method of Green forms as well.

Possible extensions of this work include propagation in anisotropic media, by treating the material properties as metrics and thereby embedding the permittivity and permeability tensors in the star operator and interior derivative.

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