Problem 3.45 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.45(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

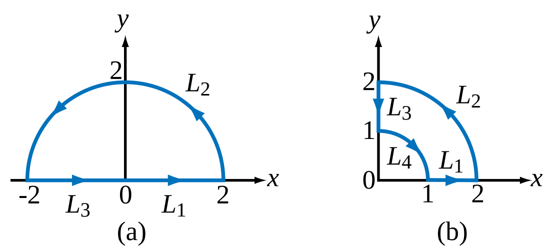

Figure P3.45: Contour paths for (a) Problem 3.45 and (b) Problem 3.46.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1}\mathbf{B}\cdot d\mathbf{l} + \int_{L_2}\mathbf{B}\cdot d\mathbf{l} + \int_{L_3}\mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=0}^{2} + 0 = 2,$$

$$\int_{L_2}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3}\mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\Big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$
\begin{aligned}
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{r}}r\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}(\sin\phi)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi))-\frac{\partial}{\partial\phi}(r\cos\phi)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}(\sin\phi+(r\sin\phi))=\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),\\
\iint\nabla\times\mathbf{B}\cdot d\mathbf{s} &= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\\
&= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi=\left(\left(-\cos\phi(\tfrac{1}{2}r^2+r)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi}=8.
\end{aligned}
$$