Electromagnetics Made Easy: Differential Forms as a Teaching Tool

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Abstract

The calculus of differential forms clarifies the relationship between field intensity and flux density and provides intuitive pictures of Ampere's and Faraday's laws, the curl operation, the Poynting vector, and boundary conditions. These and other advantages over vector analysis make differential forms an optimal tool for teaching electromagnetics.

I. Introduction

There are several areas of electromagnetic field theory which nearly every student finds difficult. Students often wonder, for example, why two vectors are required to represent a single field, or are unable to visualize the curl operation. These areas of difficulty are not fundamentally more complicated than other principles of electromagnetics, but are made obscure by the language used to express EM theory, vector analysis. There is another language for teaching electromagnetics which makes these concepts clearer and more intuitive: the calculus of differential forms.

Differential forms clarify the relationship between field intensity and flux density and provide intuitive pictures of Ampere's and Faraday's laws, the curl operation, the Poynting vector, and boundary conditions. While we have chosen to focus this paper on the visual advantages of differential forms, the notation also eliminates much of the memorization of identities, derivative formulas, and theorems required for students to compute using vector analysis. Deschamps [1] was among the first to suggest the use of differential forms as a teaching tool in engineering; Burke [2] is an active advocate in the physics community. Our own classroom experience not only supports but has helped us to refine the points we hope to make in this paper, that the calculus of differential forms makes electromagnetic field theory easier for students to visualize, understand, and apply.

II. Field Intensity and Flux Density

Associated with the electric field intensity vector $\mathbf{E}$ is the flux density vector $\mathbf{D} - \epsilon\mathbf{E}$. Graphically, $\mathbf{D}$ does not add to a student's understanding of the nature of the electric field, since the vectors differ only by a scale factor. One might justify the existence of the extra $\mathbf{D}$ vector by noting that $\mathbf{E}$ and $\mathbf{D}$ are not parallel in an anisotropic media, but there is a more fundamental reason than this.

In the commonly used alternative graphical representation of a vector field, the spacing between lines, rather than their length, represents the strength of a field. This picture, which has long been used to illuminate flux density, is not really a picture of a vector field—it is the picture of a differential form. The two pictures used to represent different types of vector fields hint at the true natures of field intensity and flux density. After introducing differential forms and the exterior product, we show that field intensity and flux density become differential forms of different degrees.

Differential Forms; Exterior Product

The calculus of differential forms is the calculus of quantities that can be integrated. The degree of a form is the dimension of the region over which it is integrated, so that in $\mathbf{R}^3$ there are 0-forms, 1-forms, 2-forms and 3-forms.
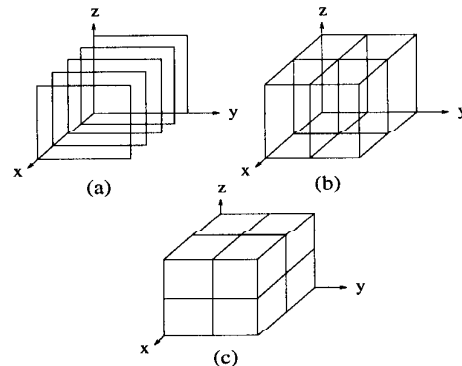
Figure 1. (a) The 1-form dx. (b) The 2-form $dx\,dy$. Tubes in the z direction are formed by the superposition of the surfaces of dx and the surfaces of dy. (c) The 3-form $dx\,dy\,dz$, with three sets of surfaces that create boxes.

1-forms are integrated over paths, and are represented graphically by surfaces [3], as in Fig. 1a. The surfaces of dx, for example, are perpendicular to the x axis, infinite in the y and z directions, and spaced a unit distance apart. The general 1-form is $E_1\,dx + E_2\,dy + E_3\,dz$, with dual vector $E_1\hat{x} + E_2\hat{y} + E_3\hat{z}$.

2-forms are integrated over surfaces. The general 2-form is $D_1\,dy \wedge dz + D_2\,dz \wedge dx + D_3\,dx \wedge dy$, with dual vector $D_1\hat{x} + D_2\hat{y} + D_3\hat{z}$. The wedge represents the exterior product, which is anticommutative, so that $dx \wedge dy = -dy \wedge dx$ and $dx \wedge dx = 0$. Wedges are often dropped for compactness. Graphically, 2-forms are tubes (Fig. 1b). The greater the coefficients of 2-form, the narrower and more dense the tubes.

A 3-form $\rho\,dx \wedge dy \wedge dz$ is a volume element, represented by boxes (Fig. 1c). The greater the magnitude of a 3-form's coefficient, the smaller and more closely spaced are the boxes. Finally, a 0-form is a function. Forms of degree greater than three vanish by the anticommutativity of the exterior product.

Because field intensities are always integrated over paths, the electric and magnetic field intensities become 1-forms, denoted by E and H. Since flux quantities are integrated over surfaces, the electric and magnetic flux densities D and B are 2-forms. Graphically, the 1-form E shows that the electric field assigns potential difference to a path. Each surface of E crossed by a path represents an increase or decrease in potential. This viewpoint on the electric field is already familiar; differential forms simply provide the mathematical framework. The 2-form D illustrates the relationship of the electric field to sources: tubes of D represent flux from positive to negative charges. The physical nature of these quantities is now encoded in the mathematical objects themselves, rather than in the choices of operators and integrals that act on them. The importance of the graphical representations for 1-forms and 2-forms will become more apparent when we show in the next section how forms allow Maxwell's laws to be visualized.

III. Maxwell's Laws

Consider the simplest example to which Ampere's law is applied, that of an infinite line current. The vector picture, shown in Fig. 2a, is often used as the fundamental example of a field with curl. The obvious question which many students encounter is, "why does the field appear to curl away from the wire?" With vectors, an imaginary "paddle wheel" must be placed in the field and an argument given as to why the wheel does not rotate. There is a much better way than Fig. 2a to visualize curl. In fact, as we will show, Ampere's law and the curl become so intuitive using differential forms that these concepts can be introduced to students first and used to motivate Gauss's law and the divergence.
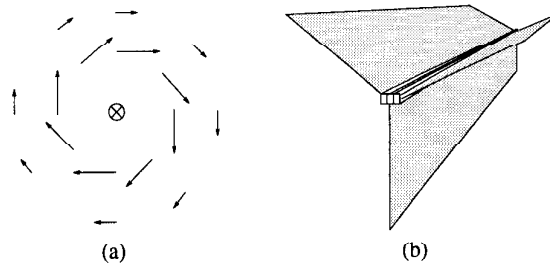

Figure 2. (a) Magnetic field intensity due to an infinite line current. (b) Ampere's law using forms: tubes of current produce magnetic field surfaces.

Maxwell's Laws Using Forms

Faraday's and Ampere's laws written using differential forms are

$$\oint_P E = -\int_A \frac{\partial}{\partial t} B, \quad \oint_P H = \int_A \left(\frac{\partial}{\partial t} D + J\right) \quad (1)$$

where P is a closed path, A is a surface bounded by P, and J is the electric current density 2-form.

Integration of forms has a simple graphical interpretation. Neglecting the obvious complications due to the orientations of differential forms and regions of integration, the integral of a 1-form over a path is simply the number of surfaces pierced by the path. The integral of a 2-form over a surface is the number of tubes passing through the surface. The integral of a 3-form over a volume is the number of boxes inside the volume.

Graphically, Ampere's law states that the number of tubes of displacement current $\frac{\partial}{\partial t} D$ and electric current J passing through a closed loop is equal to the number of surfaces of the magnetic field intensity 1-form pierced by the loop. Thus, tubes of current produce magnetic field surfaces, as illustrated in Fig. 2b.

A 1-form has nonzero curl at locations where different surfaces meet. While Fig. 2a is confusing to students because the vector field seems to rotate away from the current source, Fig. 2b shows clearly that surfaces converge only along the tubes of current, so that the field has a curl only at the source.
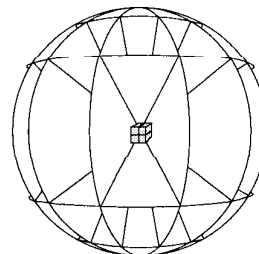

Figure 3. Gauss's law: boxes of electric charge produce tubes of electric flux.

Not only do forms clarify Ampere's and Faraday's laws, but they also elucidate the close connection between this pair of laws and Gauss's laws for the electric and magnetics fields. We write using forms,

$$\oint_S D = \int_V \rho, \quad \oint_S B = 0 \qquad (2)$$

where S is a closed surface, V is the interior of S, and ρ is the electric charge density 3-form. Graphically, the first of these laws shows that boxes of electric charge density produce tubes of electric flux density (Fig. 3). A cross section of this picture is the usual "flux" representation of the field, in which spacing between lines gives the strength of the field. Gauss's law for the magnetic field shows that tubes of magnetic flux density never end.

The concept of curl is usually much more difficult for students to grasp than that of divergence. A comparison of Figs. 2b and 3 shows that with differential forms Ampere's and Faraday's laws become as intuitive as Gauss's laws. Not only are both pairs of laws equally easy to visualize, but the conceptual unity between them is revealed.

IV. Energy and Power

The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ represents flow of power rather than intensity of a field. It is a different type of quantity than $\mathbf{E}$ or $\mathbf{H}$, just as D and E are different, and yet all these quantities have the same mathematical representation as vectors. Although it is clear from the definition that $\mathbf{S}$ is perpendicular to $\mathbf{E}$ and $\mathbf{H}$, the vector picture in Fig. 4a does not provide any intuition as to the relationship between the directions of $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{S}$. Expressing these quantities as differential forms provides added insight.
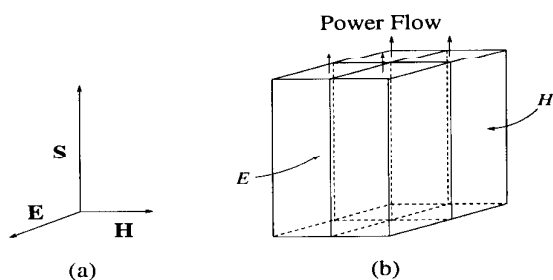

Power Flow

Figure 4. (a) The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. (b) The surfaces of the 1-forms E and H form the sides of the tubes of the Poynting power flow 2-form $S = E \wedge H$.

As Fig. 4b shows, the surfaces of the 1-forms E and H form the sides of the tubes of the Poynting 2-form S. Power flows along these tubes. This gives a clear geometrical interpretation of the fact that the direction of power flow is orthogonal to both E and H. In a similar way, the surfaces of E and H join with the tubes of D and B to form boxes of the energy density 3-form $w = \frac{1}{2}(E \wedge D + H \wedge B)$.

V. Boundary Conditions

Consider the usual vector representation of the boundary condition on the magnetic field, $\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2)$. The expressions for this and the other boundary conditions are easy to apply, but Fig. 5 has no clear physical interpretation. Differential forms provide a more appealing picture of boundary conditions.
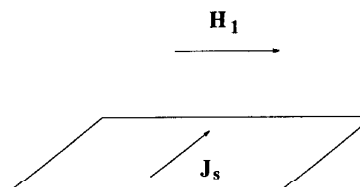

Figure 5. Surface current density $\mathbf{J}_s$ on a boundary with $\mathbf{H}_1 = \hat{y}$ above and $\mathbf{H}_2 = 0$ below.

A. The Interior Product

The interior product of 1-forms is defined (in rectangular coordinates) by $dx \lrcorner dx = dy \lrcorner dy = dz \lrcorner dz = 1$. Other combinations, such as $dx \lrcorner dy$, yield zero. For 1-forms and 2-forms,

$$dz \lrcorner (dz \wedge dx) = -dy \lrcorner (dx \wedge dy) = dx$$
$$dx \lrcorner (dx \wedge dy) = -dz \lrcorner (dy \wedge dz) = dy$$
$$dy \lrcorner (dy \wedge dz) = -dx \lrcorner (dz \wedge dx) = dz$$

and $dx \lrcorner (dy \wedge dz) = dy \lrcorner (dz \wedge dx) = dz \lrcorner (dx \wedge dy) = 0$. Graphically, the interior product removes the surfaces of the form on the right of the product from those of the form on the left.

B. Boundary Conditions Using Forms

Boundary conditions on the electromagnetic field can be written using the operator $n \lrcorner n \wedge$, where n is the 1-form $f_x dx + f_y dy + f_z dz$ normalized so that $n \lrcorner n = 1$ and f is a function that vanishes along a boundary. As proved in [4],

$$n \lrcorner (n \wedge [E_1 - E_2]) = 0$$
$$n \lrcorner (n \wedge [H_1 - H_2]) = J_s$$
$$n \lrcorner (n \wedge [D_1 - D_2]) = \rho_s$$
$$n \lrcorner (n \wedge [B_1 - B_2]) = 0$$

where subscripts represent values above ($f > 0$) and below ($f < 0$) the boundary, J_s is the surface current density 1-form, and ρ_s is the surface charge density 2-form. All the expressions use the same operator $n \lrcorner n \wedge$.

The physical difference between field intensity and flux density boundary conditions is no longer contained in the choice of cross product versus dot product, but in the degrees of the forms used to represent the field quantities.

These expressions for boundary conditions have a simple geometric interpretation. The discontinuity $H_1 - H_2$, for example, is a 1-form with surfaces that intersect the boundary along the lines of the 1-form J_s (Fig. 6a). From the fields above and below a boundary one knows immediately what sources lie on the boundary.

In the expression for J_s, the exterior product $n \wedge (H_1 - H_2)$ creates tubes with sides perpendicular to the boundary (Fig. 6b). The interior product $n \lrcorner (n \wedge [H_1 - H_2])$ removes the surfaces parallel to the boundary that were added by the exterior product, as shown in Fig. 6c. The total effect of the operator $n \lrcorner n \wedge$ is to select the component of $H_1 - H_2$ with surfaces perpendicular to the boundary.
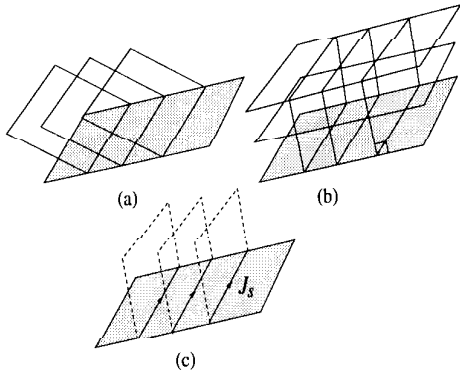

Figure 6. (a) The field discontinuity $H_1 - H_2$, which has the same intersection with the boundary as J_s. (b) The exterior product $n \wedge [H_1 - H_2]$ yields tubes with sides perpendicular to the boundary. (c) The interior product with n removes the surfaces parallel to the boundary, leaving surfaces that intersect the boundary along the lines representing the 1-form J_s.

The 1-form J_s is natural both mathematically and geometrically as a representation of surface current density. The expression for current through a path P is

$$I = \int_P \mathbf{J_s} \cdot (\hat{n} \times d\hat{s}) \qquad (3)$$

where $\hat{n}$ is a surface normal and $\hat{s}$ is tangent to the path. This simplifies to

$$I = \int_P J_s \qquad (4)$$

in terms of the 1-form J_s. We will not discuss in detail the remaining boundary conditions, but they have similar advantages over their vector counterparts.

VI. Conclusion

There are other areas of electromagnetics that are streamlined and clarified by the use of forms. We have not discussed the exterior derivative, which replaces the gradient, curl, and divergence operators and requires no memorization or table of formulas for use in curvilinear coordinates, nor the generalized Stokes' theorem, which is a single, simple relationship having the fundamental theorem of calculus, the vector Stokes theorem, and the divergence theorem as special cases. Nearly all of the common identities and formulas of vector analysis are replaced by algebraic rules that are easy for students to remember. The references, especially [5], provide more comprehensive treatments. Differential forms simplify more advanced applications, such as Green functions [6]. Ease of computation and clarity of expressions will likely extend to future applications of differential forms in applied EM theory.

In 1992, we began inserting short segments on differential forms into beginning, intermediate, and graduate EM course. Since the Fall semester of 1995, we have shifted entirely to differential forms in the beginning course. Student evaluations have been nearly unanimously positive. The most common response has been that pictures of forms help students understand electromagnetics. Ideally, preparatory calculus and physics courses would also employ differential forms. The simple correspondence between forms and vectors allows forms to be taught with little loss in understanding of materials using traditional methods.

There are other formalisms for electromagnetics: bivectors, tensors, quaternions, spinors, higher Clifford algebras, and so on. None of these offer as optimal a combination of clear relationship to traditional vector analysis, ease of presentation, concreteness, and intuitive graphical representation as differential forms. In light of the simple relationship between differential forms and vectors, establishing the calculus of differential forms as the primary language for electromagnetics would not only be desirable but feasible as well.

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